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On Power Tower of Integers

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ABSTRACT :*The power tower of an integer is defined as the iterated exponentiation of the integer, and is also called a tetration. If it is iterated for n times, it is called a power tower of order n. Using the technique of congruence, we find the pattern of the digital root, along with the last digit, of a power tower when its order increases.*

KEYWORDS:*Digital root, Power tower, Tetration*

2010MSC:*00A08, 11Y55*

I. INTRODUCTION

For a positive realnumber a, the numbers $a, a^2, a^3, \cdots$ are called the powers of a. The numbers $a, a^a, a^{a^a}, \cdots$ are called the power towersof a, or tetrations of a.However, the iterated exponential form seems not very clearly well-defined. So here we shall clarify the way how to evaluate a power tower first. The form a^{b^c} shall be operated as $a^{(b^c)}$ instead of $(a^b)^c$. That means we will operate it from the very top of the exponent downward to the base. For example, $2^{2^{2^2}}$ will be operated as $2^{\left(2^{(2^2)}\right)} = 2^{16} = 65536$, and not as $((2^2)^2)^2 = 2^8 = 256$.The power tower $\underbrace{a^{\cdot^{\cdot^{a}}}}_{n}$ with $n$iterated a's (including the base) is said to be tetrated to the order n. Historically, there have been several different notations for the power tower of a of order n. In [2] Beardon denoted it by $_na$, in [8] Knuth used the notation $(a \uparrow\uparrow n)$, and in [10] Maurer introduced the notation ^na.Maurer's notation was later adopted by Goodsteinin [6], and then popularized by Rucker (see [11]). So in this paper, we will also use Maurer's notation. Using the last example, we will denote $2^{2^{2^2}}$ by ^{42}.The study of power towers can be long traced back to Euler's work (see [5]).However at that time, and actually later on in many other studies, the focuswas on the convergence of a power tower of infinite order. In this paper, we will focus on the area more towards to recreational mathematics. We will study the last digit and the digital root of a power tower of a positive integer, and find a pattern when the order increases. For a more general knowledge of power towers, interested reader may see [11].

The numbers we are using now are based on a base-10 system. Any number can be expended to a sum of powers of 10 with coefficients less than 10. For example, the expended form of $1234$is $1 \cdot 10^3 + 2 \cdot 10^2 + 3 \cdot 10^1 + 4$. Therefore, the last coefficient in the expended form in descending order is the last digit of the number. For our convenience, we will use $ld(n)$ in this paper to indicate the last digit of the number n.

Proposition 1.1.*For any natural number n,$ld(n) = a$ where $0 \le a \le 9$ and $a \equiv n \ (mod\ 10)$.*

Proof. Consider theexpended form of n. Since $n = \sum_{i=0}^{k} a_i \cdot 10^i = 10 \cdot m + a_0$ for some k, m, and all $a_i \le 9$, we have $n \equiv a_0 \ (mod\ 10)$. By definition, $ld(n) = a_0$, hence proving the proposition.■

We know that the expended form for any number in base 10 is unique.Therefore, the last digit of any number is also unique by its definition. For other properties of the last digits not listed here, we refer to [2].

Given a number, say 1234, we will get a new (but smaller) number by adding all digits together. In the example, $1 + 2 + 3 + 4 = 10$. If the new number is not a one-digit number, we do it again and add all digits of the new number together. Repeating this process we will definitely reach a one digit number eventually. Back to our example, $1 + 0 = 1$. That one-digit number is called the digital root of the original number. We will use $dr(n)$to denote the digital root of n. Therefore, $dr(1234) = 1$. Apparently, a digital root has to be positive and it can never be zero, since we are (repeatedly) adding all non-negative digits together and they cannot be all zero.

Congruence can also be used to study the digital root. But, instead of base 10, we will use base 9. We notice that $10^k \equiv 1 \ (mod\ 9)$ for any non-negative integer k. Therefore we have the following alternative definition of digital root.

Proposition 1.2.(Congruence Formula for Digital Root) *For a natural numbern, either $dr(n) = 9$ if and only if$n \equiv 0\ (mod\ 9)$, or $dr(n) = a$ if and only if$n \equiv a\ (mod\ 9)$ where $a \in \mathbb{N}$ and $a < 9$.*

Proof. The "only if" part is trivial according to the definition, so we will only prove the "if" part. We again consider theexpended form of n. First, $n = \sum_{i=0}^{k} a_i \cdot 10^i \equiv \sum_{i=0}^{k} a_i\ (mod\ 9)$. If $\sum_{i=0}^{k} a_i$ is not a one-digit number, it can also be expressed as an expended form $\sum_{i=0}^{l} b_i \cdot 10^i$. Repeating the process, we have $\sum_{i=0}^{l} b_i \cdot$ *10i≡i=0lbi (mod 9)*. Notice that every time we repeat the process we get a smaller natural number. Eventually we will reach a one-digit number, which is still congruent to $n$modulo 9. Since in each step, the coefficients of the expended form cannot be all zero, this final number cannot be zero. So it ranges from 1to 9. If this number is less than 9, it is the a in the proposition statement. By the nature of how we find a, it is the digital root by definition. However, if $n \equiv 0\ (mod\ 9)$, the last final number in the above process can only be 9 since it's the only non-zero one-digit number that is congruent to 0 modulo 9. Therefore by definition, the digital root is9. ■

The uniqueness of digital root is not as clear as the one of last digit. So here we will provide a proof for uniqueness.

Proposition 1.3.*For any natural numbern, its digital root $dr(n)$ is unique.*

Proof. Assume that $dr(n)$ is not unique, there must be at least two distinct natural numbers a and $b$such that $dr(n) = a$and $dr(n) = b$. Applying Proposition 1.2, we have$a - b \equiv n - n = 0\ (mod\ 9)$. That means $a = 9k + b$ for a positive integer k. However, if $b \geq 1$and $k \geq 1$, $a = 9k + b$ is not a one-digit number, which contradicts the assumption of a being a digital root. ■

For more properties about digital roots, interested readers may see [1] and [9].

II. THE LAST DIGITOF A POWER TOWER

The last digit of a number seems very trivial. But, when we encounter some huge numbers like power towers, which cannot be easily written in expended form, finding their last digits is not simple anymore and requires some mathematical background. The topic of last digit of power towers has been discussed in [2, pp.33-36]. However, many details and proofs were omitted in the discussion. So here we summarize the properties, and provide all the missing proofs. We will discuss them according to their numerical order, and use congruence as our tool.

The power tower of 1 is a trivial case. We already know that 1 raised to any power is still 1. So $ld({}^{n}1) = 1$ for any n.

To formulize the last digit of power towers of 2, we first analyze the pattern of last digit of powers of 2. We notice that the last digits of $\{2^1, 2^2, 2^3, 2^4, 2^5, 2^6, \cdots\}$ form the sequence $\{2, 4, 8, 6, 2,4, \cdots\}$, which repeats every 4 numbers. This can be proved this way. We notice that $6 \cdot 2 \equiv 2\ (mod\ 10)$, $6 \cdot 4 \equiv 4\ (mod\ 10)$, $6 \cdot 8 \equiv 8\ (mod\ 10)$, and $6 \cdot 6 \equiv 6\ (mod\ 10)$ .So, $2^{4k+i} = 2^{4k} \cdot 2^i \equiv 6^k \cdot 2^i \equiv 6 \cdot 2^i \equiv 2^i\ (mod\ 10)$, where $k \in \mathbb{N}$ and $i \in \{1,2,3\}$. The one we are particularly interested is the special case that $ld(2^{4k}) = 6$ for any natural number k. Since ${}^{n}2 = 2^{({}^{n-1}2)}$ and $4|({}^{n-1}2)$for $n \geq 3$, $ld({}^{n}2) = 6$ for $n \geq 3$. As for the cases $n = 1$and $n = 2$, ${}^{1}2$ and ${}^{2}2$ are the only one-digit numbers in the power towers of 2, we can easily find their last digits, which are 2 and 4 respectively.

We still start with the powers of 3. The last digits of powers of 3 form the sequence $\{3,9,7,1,3,9, \cdots\}$. We notice that it is another 4-number repeating sequence. This time the key element is $3^4 = 81 \equiv 1\ (mod\ 10)$. Therefore, $3^{4k+i} = 3^{4k} \cdot 3^i \equiv 1 \cdot 3^i \equiv 3^i\ (mod\ 10)$, where $k \in \mathbb{N}$ and $i \in \{1,2,3\}$. In the sequence of power towers of 3, ${}^{1}3 = 3$ and ${}^{2}3 = 27$, so their last digits are 3 and 7 respectively. For $ld({}^{n}3)$when $n \geq 3$, we shall prove that it remains constant7 no matter what n is. We first notice that $3 \equiv (-1)\ (mod 4)$, and ${}^{n-2}3$ is odd. That means, ${}^{n-1}3 = 3^{({}^{n-2}3)} \equiv (-1)^{({}^{n-2}3)} = (-1) \equiv 3\ (mod\ 4)$. Therefore ${}^{n}3 = 3^{({}^{n-1}3)} = 3^{4k+3} = 3^{4k} \cdot 3^3 \equiv 3^3 \equiv 7\ (mod\ 10)$if $n \geq 3$.

The last digits of the powers of 4 form a 2-number repeating sequence $\{4,6,4,6, \cdots\}$. Since $ld(4) =$ 4and $ld(4^2) = 6$, we can easily verify that $ld(4^{2k}) = ld(6^k) = 6$ and $ld(4^{2k+1}) = ld(4^{2k} \cdot 4) = ld(6 \cdot 4) = 4$. As for power towers of 4, apparently${}^{n-1}4$ is even when $n \geq 2$. Therefore ${}^{n}4 = 4^{({}^{n-1}4)} = 4^{2k} \equiv 6\ (mod\ 10)$.

The power towers of 5 and 6 arealso trivial cases. Since $5^k \equiv 5\ (mod\ 10)$, and $6^k \equiv 6\ (mod\ 10)$, $ld({}^{n}5) = 5$ and $ld({}^{n}6) = 6$ for any $n \in \mathbb{N}$.

We first notice that $7 \equiv (-3)\ (mod\ 10)$ and ${}^{n-1}7$ is odd for any $n \geq 2$. That means ${}^{n}7 \equiv -\left(3^{({}^{n-1}7)}\right)$ (mod 10). We can then use the pattern of powers of 3 again to help us analyze power towers of 7.Since ${}^{n-1}7 \equiv (-1)^{({}^{n-2}7)} \equiv (-1) \equiv 3\ (mod\ 4)$, ${}^{n}7 \equiv -\left(3^{({}^{n-1}7)}\right) = -(3^{4k+3}) \equiv -(3^3) \equiv -7 \equiv 3\ (mod\ 10)$ for $n \geq 2$.

Like the analysis of power towers of 7, we will take advantage of the pattern of powers of 2 to work on power towers of 8. Because ${}^{n-1}8 = 4k$ for any $n \geq 2$, apparently an even number, ${}^{n}8 \equiv (-2)^{({}^{n-1}8)} = 2^{({}^{n-1}8)} = 2^{4k} \equiv 6\ (mod\ 10)$ for $n \geq 2$.

We take advantage of the fact that $9 \equiv (-1)\ (mod\ 10)$, and ${}^{n-1}9$ is odd for any $n \geq 2$. ${}^{n}9 \equiv (-1)^{({}^{n-1}9)} = -1^{({}^{n-1}9)} = -1 \equiv 9\ (mod\ 10)$.

We now summarize all the results we just derived.

Result2.1.*The sequences of last digits of power towers of integers have the following patterns:*
(i) $ld({}^{n}1) = 1$*for any natural number* n.
(ii) $ld({}^{1}2) = 2$, $ld({}^{2}2) = 4$, *and* $ld({}^{n}2) = 6$ *for any natural number* $n \geq 3$.
(iii) $ld({}^{1}3) = 3$, *and* $ld({}^{n}3) = 7$ *for any natural number* $n \geq 2$.
(iv) $ld({}^{1}4) = 4$, *and* $ld({}^{n}4) = 6$ *for any natural number* $n \geq 2$.
(v) $ld({}^{n}5) = 5$*for any natural number* n.
(vi) $ld({}^{n}6) = 6$*for any natural number* n.
(vii) $ld({}^{1}7) = 7$, *and* $ld({}^{n}7) = 3$ *for any natural number* $n \geq 2$.
(viii) $ld({}^{1}8) = 8$, *and* $ld({}^{n}8) = 6$ *for any natural number* $n \geq 2$.
(ix) $ld({}^{n}9) = 9$*for any natural number* n.

III. THE DIGITAL ROOT OF A POWER TOWER

Because the value of a power tower increases dramatically as n increases, finding its digital root is a lot more difficult than finding the last digit. As in the last section, we will still analyze the sequences of power towers according to the numerical order, but we will split some discussions into two or three paragraphs, and prove some claims in the midst of the discussion if necessary. As you may expect, congruence is still our main tool.

The power tower of 1 is still a trivial case when talking about the digital root. Since ${}^{n}1 = 1$ for any natural number n, $dr({}^{n}1) = 1$ for any natural number n.

For power towers of 2, we first handle the two simple cases. ${}^{1}2 = 2$, so $dr({}^{1}2) = 2$. ${}^{2}2 = 4$, so $dr({}^{2}2) = 4$. To find the pattern of the rest of the digital roots of the power towers of 2, we will split the discussion to two steps.

Claim 3.1.*For any natural numberk,* $2^{(2^4)^k} \equiv 7\ (mod\ 9)$.

Proof. We know that $2^{(2^4)} = 2^{16} = 65536 \equiv 7\ (mod\ 9)$ .We also notice that $2^{(2^4)^k} = 2^{(2^4)(2^4)^{k-1}} \equiv 7^{(2^4)^{k-1}} \equiv (-2)^{(2^4)^{k-1}} = 2^{(2^4)^{k-1}}\ (mod\ 9)$.Since$k$ is a finite number, repeating the process we then have $2^{(2^4)^k} \equiv 2^{(2^4)^{k-1}} \equiv 2^{(2^4)^{k-2}} \equiv \cdots \equiv 2^{(2^4)} \equiv 7\ (mod\ 9)$. ■

Claim 3.2.*For any natural numbern* ≥ 3, $dr({}^{n}2) = 7$.

Proof. If $n = 3$, ${}^{3}2 = 2^{2^2} = 16 \equiv 7\ (mod\ 9)$. If $n \geq 4$, since ${}^{n}2 = 2^{({}^{n-1}2)}$, we want to prove that ${}^{n-1}2 = (2^4)^k$ for some integer k,which is equivalent to proving ${}^{n-2}2 = 4k$. This last statement is apparently true because $4|{}^{n-2}2$if $n \geq 4$. Therefore, when $n \geq 4$, ${}^{n}2 \equiv 7\ (mod\ 9)$ according to Claim 3.1. Together with the case when$n = 3$, we proved that $dr({}^{n}2) = 7$ when $n \geq 3$.■

Since ${}^{1}3 = 3$, $dr({}^{1}3) = 3$. If $n \geq 2$, we will prove that $dr({}^{n}3) = 9$.

Claim 3.3.*For any natural number* $n \geq 2$, $dr({}^{n}3) = 9$.

Proof. We first consider the powers of 3. We notice that $9|3^k$ if $k \geq 2$. Since ${}^{n}3 = 3^{({}^{n-1}3)}$, and ${}^{n-1}3 \geq 2$ when $n \geq 2$, we easily derive that $9|{}^{n}3$ when $n \geq 2$. Therefore, ${}^{n}3 = 9k \equiv 0\ (mod\ 9)$, when $n \geq 2$. Applying proposition 1.2,$dr({}^{n}3) = 9$ when $n \geq 2$. ■

Since ${}^{1}4 = 4$, $dr({}^{1}4) = 4$. If $n \geq 2$, we will prove that $dr({}^{n}4) = 4$ as well.

Claim 3.4. *For any natural number k, $4^{4^k} \equiv 4 \ (mod\ 9)$.*

Proof. We will use mathematical induction to prove this claim. If $k = 1$, $4^{4^1} = 238 \equiv 4 \ (mod\ 9)$. Assume that $4^{4^i} \equiv 4 \ (mod\ 9)$ for an arbitrary but fixed number i. Then $4^{4^{i+1}} = 4^{4^i \cdot 4} \equiv 4^4 \equiv 4 \ (mod\ 9)$. According to the principle of mathematical induction, $4^{4^k} \equiv 4$ (mod 9) for any natural number k. ■

Claim 3.5. *For any natural number $n \geq 2$, $dr({}^{n}4) = 4$.*

Proof. Since ${}^{n}4 = 4^{({}^{n-1}4)}$, and apparently ${}^{n-1}4 = 4^k$ for some natural number k if $n \geq 2$, this claim follows immediately after Claim 3.4. ■

For the digital roots of power towers of 5, we will use a similar approach. Since ${}^{1}5 = 5$, $dr({}^{1}5) = 5$. If $n \geq 2$, we will prove that $dr({}^{n}5) = 2$.

Claim 3.6. *For any odd natural number k, $5^{5^k} \equiv 2 \ (mod\ 9)$.*

Proof. We will still use mathematical induction to prove this claim. If $k = 1$, $5^{5^1} = 3125 \equiv 2 \ (mod\ 9)$. Assume that $5^{5^{2i+1}} \equiv 2 \ (mod\ 9)$ for an arbitrary but fixed number $i \geq 1$. Then $5^{5^{2i+3}} = 5^{5^{2i+1} \cdot 5^2} \equiv 2^{5^2} = 32^5 \equiv 5^5 \equiv 2 \ (mod\ 9)$. According to the principle of mathematical induction, $5^{5^k} \equiv 2$ (mod 9) for any odd natural number k. ■

Claim 3.7. *For any natural number $n \geq 2$, $dr({}^{n}5) = 2$.*

Proof. Since ${}^{n}5 = 5^{({}^{n-1}5)}$, and apparently ${}^{n-1}5 = 5^k$ for some odd natural number k if $n \geq 2$, this claim follows immediately from Claim 3.6. ■

As usual, we discuss the one-digit element of power towers of 6 first. Since ${}^{1}6 = 6$, $dr({}^{1}6) = 6$. We then prove the rest cases.

Claim 3.8. *For any natural number $n \geq 2$, $dr({}^{n}6) = 9$.*

Proof. Since 6^k is an even number, ${}^{n}6 = 6^{({}^{n-1}6)} \equiv (-3)^{({}^{n-1}6)} = 3^{({}^{n-1}6)} \ (mod\ 9)$. Also, if $n \geq 2$, ${}^{n-1}6 > 2$. So $3^{({}^{n-1}6)} = 9k$ for some k. In other words, $3^{({}^{n-1}6)} \equiv 0 \ (mod\ 9)$, proving that $dr({}^{n}6) = 9$ by proposition 1.2. ■

Because $7 \equiv (-2) \ (mod\ 9)$, we will use a similar approach to handle power towers of 7 as we handle power towers of 2. First case, $dr({}^{1}7) = dr(7) = 7$. Second case, ${}^{2}7 = 7^7 \equiv (-2)^7 = -2^7 = -128 \equiv 7 \ (mod\ 9)$. We then have the next claim.

Claim 3.9. *For any natural number k, $7^{7^k} \equiv 7 \ (mod\ 9)$.*

Proof. Since $7^{7^k} = (7^7)^{7^{k-1}} \equiv 7^{7^{k-1}} = (7^7)^{7^{k-2}} \equiv 7^{7^{k-2}} (mod\ 9)$, repeating the process we then have $7^{7^k} \equiv 7^{7^{k-1}} \equiv 7^{7^{k-2}} \equiv \cdots \equiv 7^7 \equiv 7 \ (mod\ 9)$. ■

Claim 3.10. *For any natural number $n \geq 3$, $dr({}^{n}7) = 7$.*

Proof. Since ${}^{n}7 = 7^{({}^{n-1}7)}$, and apparently ${}^{n-1}7 = 7^k$ for some natural number k if $n \geq 3$, this claim follows immediately from Claim 3.9. ■

The power tower of 8 case is not trivial, but is relatively easy. First, $dr({}^{1}8) = dr(8) = 8$. For ${}^{n}8$ when $n \geq 2$, we have ${}^{n}8 \equiv (-1)^{({}^{n-1}8)} = 1^{({}^{n-1}8)} = 1 \ (mod\ 9)$ since ${}^{n-1}8$ is even.

The power tower of 9 is another trivial case for digital root. Since $9 \equiv 0 \ (mod\ 9)$, ${}^{n}9 = 9^{({}^{n-1}9)} \equiv 0^{({}^{n-1}9)} = 0 \ (mod\ 9)$. Therefore $dr({}^{n}9) = 9$ for any natural number n.

Summing all the above, we have the next main result.

Result 3.11. *The sequences of digital roots of power towers of integers have the following patterns:*

(i) $dr({}^{n}1) = 1$ *for any natural number n.*

(ii) $dr({}^{1}2) = 2$, $dr({}^{2}2) = 4$, *and* $dr({}^{n}2) = 7$ *for any natural number $n \geq 3$.*

(iii) $dr({}^{1}3) = 3$, *and* $dr({}^{n}3) = 9$ *for any natural number $n \geq 2$.*

(iv) $dr({}^{n}4) = 4$ *for any natural number n.*

(v) $dr({}^{1}5) = 5$, *and* $dr({}^{n}5) = 2$ *for any natural number $n \geq 2$.*

(vi) $dr(\,^{1}6) = 6$, *and* $dr(\,^{n}6) = 9$ *for any natural number* $n \geq 2$.
(vii) $dr(\,^{n}7) = 7$ *for any natural number* n.
(viii) $dr(\,^{1}8) = 8$, *and* $dr(\,^{n}8) = 1$ *for any natural number* $n \geq 2$.
(ix) $dr(\,^{n}9) = 9$ *for any natural number* n.

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