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A Comparative Study of Two-Sample t-Test Under Fuzzy Environments Using Trapezoidal Fuzzy Numbers

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ABSTRACT: *This paper proposes a method for testing hypotheses over two sample t-test under fuzzy environments using trapezoidal fuzzy numbers (tfns.). In fact, trapezoidal fuzzy numbers have many advantages over triangular fuzzy numbers as they have more generalized form. Here, we have approached a new method where trapezoidal fuzzy numbers are defined in terms of alpha level of trapezoidal interval data and based on this approach, the test of hypothesis is performed. Moreover the proposed test is analysed under various types of trapezoidal fuzzy models such as Alpha Cut Interval, Membership Function, Ranking Function, Total Integral Value and Graded Mean Integration Representation. And two numerical examples have been illustrated. Finally a comparative view of all conclusions obtained from various test is given for a concrete comparative study.*

KEYWORDS: *Trapezoidal Fuzzy Numbers (tfns./TFNS.), Alpha Cut, Test of Hypothesis, Confidence Limits, Two-sample t-Test, Ranking Function, Total Integral Value (TIV), Graded Mean Integration Representation (GMIR).*

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I. INTRODUCTION

Statistical hypothesis testing is very important tool for finding decisions in practical problems. Usually, the underlying data are assumed to be precise numbers but it is much more realistic in general to consider fuzzy values which are non-precise numbers. In this case the test statistic will also yield a non-precise number. This paper presents an approach for statistical hypothesis testing on the basis of alpha cuts of (trapezoidal) fuzzy numbers. The statistical hypotheses testing under fuzzy environments has been studied by many authors using the fuzzy set theory concepts introduced by Zadeh [40]. Viertl [35] investigated some methods to construct confidence intervals and statistical tests for fuzzy data. Wu [39] proposed some approaches to construct fuzzy confidence intervals for the unknown fuzzy parameter. A new approach to the problem of testing statistical hypotheses is introduced by Chachi et al. [15]. Asady [9] introduced a method to obtain the nearest trapezoidal approximation of fuzzy numbers. Gajivaradhan and Parthiban analysed one sample t-test and two sample t-test using alpha cut interval method using trapezoidal fuzzy numbers [18, 19]. Abhinav Bansal [5] explored some arithmetic properties of arbitrary trapezoidal fuzzy numbers of the form (a, b, c, d). Moreover, Liou and Wang ranked fuzzy numbers with total integral value [25]. Wang et al. presented the method for centroid formulae for a generalized fuzzy number [37]. Iuliana Carmen BĂRBĂCIORU dealt with the statistical hypotheses testing using membership function of fuzzy numbers [21]. Salim Rezvani analysed the ranking functions with trapezoidal fuzzy numbers [30]. Wang arrived some different approach for ranking trapezoidal fuzzy numbers [37]. Thorani et al. approached the ranking function of a trapezoidal fuzzy number with some modifications [31]. Salim Rezvani and Mohammad Molani presented the shape function and Graded Mean Integration Representation (GMIR) for trapezoidal fuzzy numbers [29]. Liou and Wang proposed the Total Integral Value (TIV) of the trapezoidal fuzzy number with the index of optimism and pessimism [25].

In this paper, we propose a new statistical fuzzy hypothesis testing of two-sample t-test in which the designated samples are in terms of fuzzy (trapezoidal fuzzy numbers) data. Another idea in this paper is, when we have some vague data about an experiment, what can be the result when the centroid point/ranking grades of those imprecise data are employed in hypothesis testing? For this reason, we have used the centroid/ranking grades of trapezoidal fuzzy numbers (tfns.) in hypothesis testing. In the decision rules of the proposed testing technique, degrees of optimism, pessimism and h-level sets are not used. In fact, we would like to counter an argument that the α-cut interval method can be general enough to deal with two-sample t-test under fuzzy environments. And for better understanding, the proposed fuzzy hypothesis testing technique is illustrated with two numerical examples at each models. Finally a tabular form of all conclusions obtained from various test is given for a concrete comparative study. And the same concept can also be used when we have samples in terms of triangular fuzzy numbers [10].

II. PRELIMINARIES

Definition 2.1. Generalized fuzzy number

A generalized fuzzy number $\widetilde{A} = (a, b, c, d; w)$ is described as any fuzzy subset of the real line $\mathbb{R}$, whose membership function $\mu_{\widetilde{A}}(x)$ satisfies the following conditions:

i. $\mu_{\widetilde{A}}(x)$ is a continuous mapping from $\mathbb{R}$ to the closed interval $[0, \omega],\ 0 \le \omega \le 1$,

ii. $\mu_{\widetilde{A}}(x) = 0$, for all $x \in (-\infty, a]$,

iii. $\mu_L(x) = L_{\widetilde{A}}(x)$ is strictly increasing on $[a, b]$,

iv. $\mu_{\widetilde{A}}(x) = \omega$, for all $[b, c]$ and ω is a constant, $0 < \omega \le 1$,

v. $\mu_R(x) = R_{\widetilde{A}}(x)$ is strictly decreasing on $[c, d]$,

vi. $\mu_{\widetilde{A}}(x) = 0$, for all $x \in [d, \infty)$ where a, b, c, d are real numbers such that $a < b \le c < d$.

Definition 2.2. A fuzzy set $\widetilde{A}$ is called ***normal*** fuzzy set if there exists an element (member) 'x' such that $\mu_{\widetilde{A}}(x) = 1$. A fuzzy set $\widetilde{A}$ is called ***convex*** fuzzy set if $\mu_{\widetilde{A}}(\alpha x_1 + (1 - \alpha)x_2) \ge \min\{\mu_{\widetilde{A}}(x_1), \mu_{\widetilde{A}}(x_2)\}$ where $x_1, x_2 \in X$ and $\alpha \in [0, 1]$. The set $\widetilde{A}_\alpha = \{x \in X / \mu_{\widetilde{A}}(x) \ge \alpha\}$ is said to be the α - cut of a fuzzy set $\widetilde{A}$.

Definition 2.3. A fuzzy subset $\widetilde{A}$ of the real line $\mathbb{R}$ with ***membership function*** $\mu_{\widetilde{A}}(x)$ such that $\mu_{\widetilde{A}}(x): \mathbb{R} \to [0, 1]$, is called a fuzzy number if $\widetilde{A}$ is normal, $\widetilde{A}$ is fuzzy convex, $\mu_{\widetilde{A}}(x)$ is upper semi-continuous and $\mathrm{Supp}(\widetilde{A})$ is bounded, where $\mathrm{Supp}(\widetilde{A}) = \mathrm{cl}\{x \in \mathbb{R}:\ \mu_{\widetilde{A}}(x) > 0\}$ and 'cl' is the closure operator.

It is known that for a ***normalized tfn.*** $\widetilde{A} = (a, b, c, d; 1)$, there exists four numbers $a, b, c, d \in \mathbb{R}$ and two functions $L_{\widetilde{A}}(x), R_{\widetilde{A}}(x): \mathbb{R} \to [0, 1]$, where $L_{\widetilde{A}}(x)$ and $R_{\widetilde{A}}(x)$ are non-decreasing and non-increasing functions respectively. And its membership function is defined as follows:

$\mu_{\widetilde{A}}(x) = \{L_{\widetilde{A}}(x) = (x\text{-}a)/(b\text{-}a)$ for $a \le x \le b$; 1 for $b \le x \le c$; $R_{\widetilde{A}}(x) = (x\text{-}d)/(c\text{-}d)$ for $c \le x \le d$ and 0 otherwise}. The functions $L_{\widetilde{A}}(x)$ and $R_{\widetilde{A}}(x)$ are also called the ***left*** and ***right side*** of the fuzzy number $\widetilde{A}$ respectively [17]. In this paper, we assume that $\int_{-\infty}^{\infty} \widetilde{A}(x)dx < +\infty$ and it is known that the α - cut of a fuzzy number is $\widetilde{A}_\alpha = \{x \in \mathbb{R} / \mu_{\widetilde{A}}(x) \ge \alpha\}$, for $\alpha \in (0, 1]$ and $\widetilde{A}_0 = \mathrm{cl}\left(\bigcup_{\alpha \in (0, 1]} \widetilde{A}_\alpha\right)$, according to the definition of a fuzzy number, it is seen at once that every α - cut of a fuzzy number is a closed interval. Hence, for a fuzzy number $\widetilde{A}$, we have $\widetilde{A}(\alpha) = [\widetilde{A}_L(\alpha), \widetilde{A}_U(\alpha)]$ where $A_L(\alpha) = \inf\{x \in \mathbb{R}:\ \mu_{\widetilde{A}}(x) \ge \alpha\}$ and $A_U(\alpha) = \sup\{x \in \mathbb{R}:\ \mu_{\widetilde{A}}(x) \ge \alpha\}$. The left and right sides of the fuzzy number $\widetilde{A}$ are strictly monotone, obviously, $\widetilde{A}_L$ and $\widetilde{A}_U$ are inverse functions of $L_{\widetilde{A}}(x)$ and $R_{\widetilde{A}}(x)$ respectively. Another important type of fuzzy number was introduced in [11] as follows:

Let $a, b, c, d \in \mathbb{R}$ such that $a < b \le c < d$. A fuzzy number $\widetilde{A}$ defined as $\mu_{\widetilde{A}}(x):\ \mathbb{R} \to [0, 1]$,

$\mu_{\tilde{A}}(x) = \left(\frac{x - a}{b - a}\right)^n$ for $a \le x \le b$; 1 for $b \le x \le c$; $\left(\frac{d - x}{d - c}\right)^n$ for $c \le x \le d$; 0 otherwise where $n > 0$, is denoted by $\tilde{A} = (a, b, c, d)_n$. And $L(x) = \left(\frac{x - a}{b - a}\right)^n$ and $R(x) = \left(\frac{d - x}{d - c}\right)^n$ can also be termed as left and right spread of the tfn. [Dubois and Prade in 1981].

If $\tilde{A} = (a, b, c, d)_n$, then[1-4],

$\tilde{A}_\alpha = \left[\tilde{A}_L(\alpha),\ \tilde{A}_U(\alpha)\right] = \left[a + (b - a)\sqrt[n]{\alpha},\ d - (d - c)\sqrt[n]{\alpha}\right];\ \alpha \in [0, 1]$.

When $n = 1$ and $b = c$, we get a triangular fuzzy number. The conditions $r = 1, a = b$ and $c = d$ imply the closed interval and in the case $r = 1, a = b = c = d = t$ (some constant), we can get a crisp number 't'. Since a trapezoidal fuzzy number is completely characterized by $n = 1$ and four real numbers $a \le b \le c \le d$, it is often denoted as $\tilde{A} = (a, b, c, d)$. And the family of trapezoidal fuzzy numbers will be denoted by $F^T(\mathbb{R})$. Now, for $n = 1$ we have a normal trapezoidal fuzzy number $\tilde{A} = (a, b, c, d)$ and the corresponding α - cut is defined by

$\tilde{A}_\alpha = \left[a + \alpha(b - a),\ d - \alpha(d - c)\right];\ \alpha \in [0, 1] --- (2.4)$. And we need the following results which can be found in [22, 24].

Result 2.1. Let $D = \{[a, b],\ a \le b \text{ and } a, b \in \mathbb{R}\}$, the set of all closed, bounded intervals on the real line $\mathbb{R}$.

Result 2.2. Let $A = [a, b]$ and $B = [c, d]$ be in D. Then $A = B$ if $a = c$ and $b = d$.

III. TWO – SAMPLE (STUDENT) t-TEST

Let x_i and y_j, $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$ be two random samples from two different normal populations with sizes m and n such that $m + n \le 30$. And let $\bar{x}$ and $\bar{y}$ be the mean values and s_1 and s_2 be the sample standard deviations of the random variables x_i and y_j respectively and they are given by [21],

$$\bar{x} = \frac{1}{m}\left(\sum_{i=1}^{m} x_i\right) \text{ and } \bar{y} = \frac{1}{n}\left(\sum_{i=1}^{n} y_i\right)$$

$$s_1 = \sqrt{\left(\frac{1}{m - 1}\right)\left(\sum_{i=1}^{m}\left(x_i - \bar{x}\right)^2\right)} \text{ and } s_2 = \sqrt{\left(\frac{1}{n - 1}\right)\left(\sum_{j=1}^{n}\left(y_j - \bar{y}\right)^2\right)} --- (1)$$

Let μ_1 and μ_2 be the population means of X-sample and Y-sample respectively. In testing the null hypothesis $H_0 : \mu_1 = \mu_2$ assuming with **equal** population standard deviation, we generally use the test statistic:

$$t = \frac{\bar{x} - \bar{y}}{s\sqrt{\frac{1}{m} + \frac{1}{n}}} \quad \text{where} \quad s = \sqrt{\frac{(m - 1)s_1^2 + (n - 1)s_2^2}{m + n - 2}}$$

In testing the null hypothesis $H_0 : \mu_1 = \mu_2$ assuming with **unequal** population standard deviation, we commonly use the test statistic:

$$t = \frac{\bar{x} - \bar{y}}{\sqrt{\frac{s_1^2}{m} + \frac{s_2^2}{n}}}$$

Now, the degrees of freedom used in this test is $\nu = n + m - 2$. Let α be the level of significance and let $t_{\alpha, v}$ be the tabulated value of 't' for v degrees of freedom at α level of significance. And the null hypothesis is given by $H_0 : \mu_1 = \mu_2$. And the rejection region for the alternative hypothesis H_A at α level is given below:

Alternative Hypothesis H_A	**Rejection Region at** α **Level**
$H_A : \mu_1 > \mu_2$	$t \geq t_{\alpha, m+n-2}$ (Upper tailed test)
$H_A : \mu_1 < \mu_2$	$t \leq -t_{\alpha, m+n-2}$ (Lower tailed test)
$H_A : \mu_1 \neq \mu_2$	$\lvert t \rvert \geq t_{\alpha/2, m+n-2}$ (Two tailed test)

If $\lvert t \rvert < t_{\alpha, m+n-2}$ (one tailed test), the difference between μ_1 and μ_2 is not significant at α level. Then the means of the populations are identical. That is, $\mu_1 = \mu_2$ at α level of significance. Therefore, the null hypothesis H_0 is accepted. Otherwise, the alternative hypothesis H_A is accepted.

If $\lvert t \rvert < t_{\alpha/2, m+n-2}$ (two tailed test), the difference between μ_1 and μ_2 is not significant at α level. Then the means of the populations are identical. That is, $\mu_1 = \mu_2$ at α level of significance. Therefore, the null hypothesis H_0 is accepted. Otherwise, the alternative hypothesis H_A is accepted.

Now, the $100(1-\alpha)\%$ confidence limits for the difference of population means μ_1 and μ_2 corresponding to the given samples are given by,

$$\left(\bar{x} - \bar{y}\right) - t_{\alpha/2, m+n-2}\left(s\sqrt{\frac{1}{m}+\frac{1}{n}}\right) < \left(\mu_1 - \mu_2\right) < \left(\bar{x} + \bar{y}\right) + t_{\alpha/2, m+n-2}\left(s\sqrt{\frac{1}{m}+\frac{1}{n}}\right)$$

(for equal standard deviations) Or

$$\left(\bar{x} - \bar{y}\right) - t_{\alpha/2, m+n-2}\left(\sqrt{\frac{s_1^2}{m}+\frac{s_2^2}{n}}\right) < \left(\mu_1 - \mu_2\right) < \left(\bar{x} + \bar{y}\right) + t_{\alpha/2, m+n-2}\left(\sqrt{\frac{s_1^2}{m}+\frac{s_2^2}{n}}\right)$$

(for unequal standard deviations)

IV. TEST OF HYPOTHESIS FOR INTERVAL DATA

Let $\{[a_i, b_i],\ i = 1, 2, ..., m\}$ be a random small sample (X-sample) with size m and $\{[c_j, d_j],\ j = 1, 2, ..., n\}$ be a random small sample (Y-sample) with size n. And let $[\eta_1, \mu_1]$ be mean of X from a normal population and let $[\eta_2, \mu_2]$ be mean of Y from another normal population.

Now, we test the null hypothesis H_0 such that the means of the population of the given samples are equal. That is $H_0 : [\eta_1, \mu_1] = [\eta_2, \mu_2] \Rightarrow \eta_1 = \eta_2$ and $\mu_1 = \mu_2$. And the alternative hypotheses are given by [16],

i. $H_A : [\eta_1, \mu_1] < [\eta_2, \mu_2] \Rightarrow \eta_1 < \eta_2$ and $\mu_1 < \mu_2$

ii. $H_A : [\eta_1, \mu_1] > [\eta_2, \mu_2] \Rightarrow \eta_1 > \eta_2$ and $\mu_1 > \mu_2$

iii. $H_A : [\eta_1, \mu_1] \neq [\eta_2, \mu_2] \Rightarrow \eta_1 \neq \eta_2$ or $\mu_1 \neq \mu_2$

Now the lower values and upper values for X-sample and Y-sample are given below:

X_L (Lower values of X-sample)	a_i ; $i = 1, 2, ..., m$
Y_L (Lower values of Y-sample)	c_j ; $j = 1, 2, ..., n$
X_U (Upper values of X-sample)	b_i ; $i = 1, 2, ..., m$
Y_U (Upper values of Y-sample)	d_j ; $j = 1, 2, ..., n$

Let $\bar{x}_L$ and $\bar{y}_L$ be the sample means, s_{x_L} and s_{y_L} be the sample standard deviation of X^L and Y^L respectively. Similarly let $\bar{x}_U$ and $\bar{y}_U$ be the sample means, s_{x_U} and s_{y_U} be the sample standard deviation of X^U and Y^U respectively.

Case (i): If the population standard deviations are assumed to be **equal,** then under the null hypothesis $H_0 : [\eta_1, \mu_1] = [\eta_2, \mu_2]$, the test statistic is given by,

$$t_L = \frac{\bar{x}_L - \bar{y}_L}{s_L\sqrt{\frac{1}{m}+\frac{1}{n}}} \quad \text{and} \quad t_U = \frac{\bar{x}_U - \bar{y}_U}{s_U\sqrt{\frac{1}{m}+\frac{1}{n}}}$$

where $s_L = \sqrt{\frac{(m-1)s_{x_L}^2 + (n-1)s_{y_L}^2}{m+n-2}}$ and $s_U = \sqrt{\frac{(m-1)s_{x_U}^2 + (n-1)s_{y_U}^2}{m+n-2}}$

Case (ii): If the population standard deviations are assumed to be **unequal**, then under the null hypothesis $H_0 : [\eta_1, \mu_1] = [\eta_2, \mu_2]$, the test statistic is given by,

$$t_L = \frac{\bar{x}_L - \bar{y}_L}{\sqrt{\frac{s_{x_L}^2}{m}+\frac{s_{y_L}^2}{n}}} \quad \text{and} \quad t_U = \frac{\bar{x}_U - \bar{y}_U}{\sqrt{\frac{s_{x_U}^2}{m}+\frac{s_{y_U}^2}{n}}}$$

where the standard deviations for upper and lower values of the samples of X and Y are given by the equation (1).

And the rejection region of the alternative hypothesis H_A at α level of significance is given below:

Alternative Hypothesis H_A	**Rejection Region at α Level**
$H_A : [\eta_1, \mu_1] > [\eta_2, \mu_2]$	$t_L \geq t_{\alpha, m+n-2}$ and $t_U \geq t_{\alpha, m+n-2}$ (Upper tailed test)
$H_A : [\eta_1, \mu_1] < [\eta_2, \mu_2]$	$t_L \leq -t_{\alpha, m+n-2}$ and $t_U \leq -t_{\alpha, m+n-2}$ (Lower tailed test)
$H_A : [\eta_1, \mu_1] \neq [\eta_2, \mu_2]$	$\lvert t_L \rvert \geq t_{\alpha/2, m+n-2}$ or $\lvert t_U \rvert \geq t_{\alpha/2, m+n-2}$ (Two tailed test)

If $|t_L| < t_{\alpha, m+n-2}$ and $|t_U| < t_{\alpha, m+n-2}$ (one tailed test), then the difference between $[\eta_1, \mu_1]$ and $[\eta_2, \mu_2]$ is not significant at α level. Then the means of the populations are identical. That is, $[\eta_1, \mu_1] = [\eta_2, \mu_2]$ at α level of significance. Therefore, the null hypothesis H_0 is accepted. Otherwise, the alternative hypothesis H_A is accepted.

If $|t_L| < t_{\alpha/2, m+n-2}$ and $|t_U| < t_{\alpha/2, m+n-2}$ (two tailed test), the difference between $[\eta_1, \mu_1]$ and $[\eta_2, \mu_2]$ is not significant at α level. Then the means of the populations are identical. That is, $[\eta_1, \mu_1] = [\eta_2, \mu_2]$ at α level of significance. Therefore, the null hypothesis H_0 is accepted. Otherwise, the alternative hypothesis H_A is accepted.

And the $100(1-\alpha)\%$ confidence limits for the difference of lower limit and upper limit of the population means $[\eta_1, \mu_1]$ and $[\eta_2, \mu_2]$ corresponding to the given samples are given below:

$$(\bar{x}_L - \bar{y}_L) - t_{\alpha/2, m+n-2}\left(s_L\sqrt{\frac{1}{m}+\frac{1}{n}}\right) < (\eta_1 - \eta_2) < (\bar{x}_L - \bar{y}_L) + t_{\alpha/2, m+n-2}\left(s_L\sqrt{\frac{1}{m}+\frac{1}{n}}\right)$$

(for equal population standard deviations), and

$$\left(\bar{x}_U - \bar{y}_U\right) - t_{\alpha/2,\, m+n-2}\left(s_U\sqrt{\frac{1}{m}+\frac{1}{n}}\right) < \left(\eta_1 - \eta_2\right) < \left(\bar{x}_U - \bar{y}_U\right) + t_{\alpha/2,\, m+n-2}\left(s_U\sqrt{\frac{1}{m}+\frac{1}{n}}\right)$$

(for equal population standard deviations) Or

$$\left(\bar{x}_L - \bar{y}_L\right) - t_{\alpha/2,\, m+n-2}\left(\sqrt{\frac{s^2_{x_L}}{m}+\frac{s^2_{y_L}}{n}}\right) < \left(\eta_1 - \eta_2\right) < \left(\bar{x}_L - \bar{y}_L\right) + t_{\alpha/2,\, m+n-2}\left(\sqrt{\frac{s^2_{x_L}}{m}+\frac{s^2_{y_L}}{n}}\right)$$

(for unequal population standard deviations) and

$$\left(\bar{x}_U - \bar{y}_U\right) - t_{\alpha/2,\, m+n-2}\left(\sqrt{\frac{s^2_{x_U}}{m}+\frac{s^2_{y_U}}{n}}\right) < \left(\eta_1 - \eta_2\right) < \left(\bar{x}_U - \bar{y}_U\right) + t_{\alpha/2,\, m+n-2}\left(\sqrt{\frac{s^2_{x_U}}{m}+\frac{s^2_{y_U}}{n}}\right)$$

(for unequal population standard deviations)

Decision table:

Acceptance of null hypotheses $\widetilde{H}_0$		
Lower Level Model	**Upper Level Model**	**Conclusion**
If H_0 is accepted for all $\alpha\ \alpha \in [0,1]$	and H_0 is accepted for all $\alpha\ \alpha \in [0,1]$	then $\widetilde{H}_0$ is accepted for all $\alpha\ \alpha \in [0,1]$
If H_0 is accepted for all $\alpha\ \alpha \in [0,1]$	and H_0 is rejected for all $\alpha\ \alpha \in [0,1]$	then $\widetilde{H}_0$ is rejected for all $\alpha\ \alpha \in [0,1]$
If H_0 is rejected for all $\alpha\ \alpha \in [0,1]$	and H_0 is accepted for all $\alpha\ \alpha \in [0,1]$	then $\widetilde{H}_0$ is rejected for all $\alpha\ \alpha \in [0,1]$
If H_0 is rejected for all $\alpha\ \alpha \in [0,1]$	or H_0 is rejected for all $\alpha\ \alpha \in [0,1]$	then $\widetilde{H}_0$ is rejected for all $\alpha\ \alpha \in [0,1]$

Partial acceptance of null hypothesis H_0 at the intersection of certain level of α at both upper level and lower level models can be taken into account for the acceptance of the null hypothesis $\widetilde{H}_0$.

This test procedure has been illustrated using the following numerical examples.

Example-1

The following interval data are given the gain in weights (in lbs) of pet dogs fed on two kinds of diets A and B [21].

Diet-A	Diet-B	Diet-A	Diet-B
[18, 19]	[22, 26]	[19, 22]	[22, 28]
[16, 18]	[27, 31]	[20, 24]	[20, 24]
[30, 32]	[25, 28]	[27, 30]	[11, 15]
[28, 30]	[12, 16]	[18, 22]	[14, 17]
[22, 24]	[16, 20]	[21, 24]	[17, 21]
[14, 16]	[18, 22]	--	[25, 27]
[28, 32]	[26, 30]	--	[19, 22]
		--	[23, 25]

Now, we test if the two diets differ significantly on the basis of their nutrition effects on increase in the weight of the pet dogs.

Here the null hypothesis is, $H_0 : [\eta_1, \mu_1] = [\eta_2, \mu_2] \Rightarrow \eta_1 = \eta_2$ and $\mu_1 = \mu_2$.

$\Rightarrow$ There is no significant difference between the nutrition effects from diet A and diet B.

And the alternative hypothesis is $H_A : [\eta_1, \mu_1] \neq [\eta_2, \mu_2] \Rightarrow \eta_1 \neq \eta_2$ or $\mu_1 \neq \mu_2$ (Two tailed test).

$\Rightarrow$ The two kinds of the diets differ significantly on the basis of their nutrition effects.

We assume that the **standard deviations of the populations are not equal** and we use 5% level of significance. Here m=12 and n=15.

The tabulated value of 't' for m + n – 2 = 27 – 2 = 25 degrees of freedom at 5% level of significance is $T_\alpha = 2.06$

Now, $\bar{x}_L = \frac{1}{m}\left(\sum_{i=1}^{m} x_{i_L}\right) \Rightarrow \bar{x}_L = 21.75$ and $\bar{y}_L = \frac{1}{n}\left(\sum_{i=1}^{n} y_{i_L}\right) \Rightarrow \bar{y}_L = 20.0667$ and

$\bar{x}_U = \frac{1}{m}\left(\sum_{i=1}^{m} x_{i_U}\right) \Rightarrow \bar{x}_U = 24.4167$ and $\bar{y}_U = \frac{1}{n}\left(\sum_{i=1}^{n} y_{i_U}\right) \Rightarrow \bar{y}_U = 23.7333$

$s^2_{x_L} = \left(\frac{1}{m-1}\right)\sum_{i=1}^{m}\left(x_{i_L} - \bar{x}_L\right)^2 \Rightarrow s^2_{x_L} = 27.8409$ and $s^2_{y_L} = \left(\frac{1}{n-1}\right)\sum_{i=1}^{n}\left(y_{i_L} - \bar{y}_L\right)^2 \Rightarrow s^2_{y_L} = 27.0667$

$s^2_{x_U} = \left(\frac{1}{m-1}\right)\sum_{i=1}^{m}\left(x_{i_U} - \bar{x}_U\right)^2 \Rightarrow s^2_{x_U} = 30.0833$ and $s^2_{y_U} = \left(\frac{1}{n-1}\right)\sum_{i=1}^{n}\left(y_{i_U} - \bar{y}_U\right)^2 \Rightarrow s^2_{y_U} = 26.6381$

The Test Statistics:

$t_L = \frac{\bar{x}_L - \bar{y}_L}{\sqrt{\frac{s^2_{x_L}}{m} + \frac{s^2_{y_L}}{n}}} \Rightarrow t_L = 0.8288$ and $t_U = \frac{\bar{x}_U - \bar{y}_U}{\sqrt{\frac{s^2_{x_U}}{m} + \frac{s^2_{y_U}}{n}}} \Rightarrow t_U = 0.3302$

Since, $|t_L| < T_\alpha = 2.06$ and $|t_U| < T_\alpha = 2.06$, we accept the null hypothesis H_0.

$\Rightarrow$ There is no significant difference between the nutrition effects of the diets A and B at 5% level of significance.

V. TEST OF HYPOTHESIS FOR FUZZY DATA USING TFNS.

Definition 5.1: Trapezoidal Fuzzy Number to Interval

Let a trapezoidal fuzzy number be defined as $\widetilde{A} = (a, b, c, d)$, then the fuzzy interval in terms of α-cut interval is defined as follows [33]:

$$\widetilde{A}_\alpha = \left[a + (b - a)\alpha, d - (d - c)\alpha\right]; 0 \le \alpha \le 1 \quad ---(5.1)$$

Suppose that the given sample is a fuzzy data that are trapezoidal fuzzy numbers and we have to test the hypothesis about the population mean. Using the relation (1) and the proposed test procedure, we can test the hypothesis by transferring the fuzzy data into interval data.

Example-2

Two kinds of engine oils A and B for automobiles are under mileage test for some taxies, then we request the taxi drivers to record the consumption of fuel. Due to limited available source, the data are recorded as trapezoidal fuzzy numbers which are given in the following table. Suppose the random variables have normal distribution and their variances of both populations are **known and equal with one**. We now investigate the effects of the two kinds of engine oils on consumption of fuel at 5% level of significance [10].

$\widetilde{A}$	$\widetilde{B}$
(4, 4.5, 5, 6)	(5, 6.5, 7, 8)
(3.5, 4, 5, 6.5)	(4, 4.5, 5, 6)
(5, 5.5, 5.8, 6)	(5.5, 7, 8, 8.5)
(5.5, 5.8, 6, 6.5)	(5, 6, 6.5, 7)
(3, 3.5, 4, 5)	(6, 6.5, 7, 8)
--	(6, 7.5, 8.5, 9)

Now the interval representation of the above trapezoidal data is given below:

$\left[\widetilde{A}_\alpha\right]$	$\left[\widetilde{B}_\alpha\right]$
$[4 + 0.5\alpha, 6 - \alpha]$	$[5 + 1.5\alpha, 8 - \alpha]$
$[3.5 + 0.5\alpha, 6.5 - 1.5\alpha]$	$[4 + 0.5\alpha, 6 - \alpha]$
$[5 + 0.5\alpha, 6 - 0.2\alpha]$	$[5.5 + 1.5\alpha, 8.5 - 0.5\alpha]$
$[5.5 + 0.3\alpha, 6.5 - 0.5\alpha]$	$[5 + \alpha, 7 - 0.5\alpha]$
$[3 + 0.5\alpha, 5 - \alpha]$	$[6 + 0.5\alpha, 8 - \alpha]$
--	$[6 + 1.5\alpha, 9 - 0.5\alpha]$

Lower Level Samples		**Upper Level Samples**	
x_L	y_L	x_U	y_U
4 + 0.5α	5 + 1.5α	6 - α	8 - α
3.5 + 0.5α	4 + 0.5α	6.5 - 1.5α	6 - α
5 + 0.5α	5.5 + 1.5α	6 - 0.2α	8.5 - 0.5α
5.5 + 0.3α	5 + α	6.5 - 0.5α	7 - 0.5α
3 + 0.5α	6 + 0.5α	5 - α	8 - α
--	6 + 1.5α	--	9 - 0.5α

Here, m = 5 and n = 6.

$$\bar{x}_L = \frac{1}{m}\left(\sum_{i=1}^{m} x_{i_L}\right) \Rightarrow \bar{x}_L = 4.2 + 0.46\alpha \text{ and } \bar{x}_U = \frac{1}{m}\left(\sum_{i=1}^{m} x_{i_U}\right) \Rightarrow \bar{x}_U = 6 - 0.84\alpha$$

$$\bar{y}_L = \frac{1}{n}\left(\sum_{i=1}^{n} y_{i_L}\right) \Rightarrow \bar{y}_L = 5.25 + 1.083\alpha \text{ and } \bar{y}_U = \frac{1}{n}\left(\sum_{i=1}^{n} y_{i_U}\right) \Rightarrow \bar{y}_U = 7.75 - 0.75\alpha$$

$$s^2_{x_L} = \left(\frac{1}{m-1}\right)\sum_{i=1}^{m}\left(x_{i_L} - \bar{x}_L\right)^2 = 0.008\alpha^2 - 0.13\alpha + 1.075 \text{ and}$$

$$s^2_{y_L} = \left(\frac{1}{n-1}\right)\sum_{i=1}^{n}\left(y_{i_L} - \bar{y}_L\right)^2 = 0.2417\alpha^2 + 0.25\alpha + 0.5750$$

$$s^2_{x_U} = \left(\frac{1}{m-1}\right)\sum_{i=1}^{m}\left(x_{i_U} - \bar{x}_U\right)^2 = 0.2402\alpha^2 + 0.375 \text{ and}$$

$$s^2_{y_U} = \left(\frac{1}{n-1}\right)\sum_{i=1}^{n}\left(y_{i_U} - \bar{y}_U\right)^2 = 0.05\alpha^2 + 0.25\alpha - 0.05 \text{ and}$$

$$S^2_L = \frac{(m-1)s^2_{x_L} + (n-1)s^2_{y_L}}{(m+n-2)} = 0.1379\alpha^2 - 0.1967\alpha + 0.7972$$

$$S^2_U = \frac{(m-1)s^2_{x_U} + (n-1)s^2_{y_U}}{(m+n-2)} = 0.1346\alpha^2 + 0.1389\alpha + 0.1389$$

Null hypothesis:

$\widetilde{H}_0 : \widetilde{\Lambda} \approx \widetilde{\Omega} \Rightarrow$ The two kinds of engine oils on fuel consumption are same.

Alternative hypothesis:

$\widetilde{H}_A : \widetilde{\Lambda} \neq \widetilde{\Omega} \Rightarrow$ The two kinds of engine oils on fuel consumption differ significantly.

Here, $\left[\widetilde{\Lambda}\right]=\left[\eta_1,\mu_1\right]$ and $\left[\widetilde{\Omega}\right]=\left[\eta_2,\mu_2\right]$.

And therefore, $\left[\widetilde{H}_0\right]:\left[\widetilde{\Lambda}\right] \approx \left[\widetilde{\Omega}\right] \Rightarrow \widetilde{H}_0:\eta_1=\eta_2$ and $\mu_1=\mu_2$.

$\left[\widetilde{H}_A\right]:\left[\widetilde{\Lambda}\right] \neq \left[\widetilde{\Omega}\right] \Rightarrow \widetilde{H}_A:\eta_1 \neq \eta_2$ or $\mu_1 \neq \mu_2$ (Two tailed test).

Now, the tabulated value of 't' at 5% level of significance with 9 degrees of freedom is $T_\alpha = 2.262$

Test statistics:

$$t_L = \frac{\bar{x}_L - \bar{y}_L}{\sqrt{\frac{s_{x_L}^2}{m}+\frac{s_{y_L}^2}{n}}} = \begin{cases} -1.9422 & \text{if } \alpha = 0 \\ -2.0814 & \text{if } \alpha = 0.1 \\ -2.2204 & \text{if } \alpha = 0.2 \\ -2.3578 & \text{if } \alpha = 0.3 \\ \cdots & \\ -3.2154 & \text{if } \alpha = 1 \end{cases} \qquad \Rightarrow |t_L| > T_\alpha \text{ for } 0.3 \le \alpha \le 1$$

$$t_U = \frac{\bar{x}_U - \bar{y}_U}{\sqrt{\frac{s_{x_U}^2}{m}+\frac{s_{y_U}^2}{n}}} = \begin{cases} -7.7537 & \text{if } \alpha = 0 \\ -4.7320 & \text{if } \alpha = 1 \end{cases} \qquad \Rightarrow |t_U| > T_\alpha \text{ for } 0 \le \alpha \le 1$$

Conclusion

Hence, $|t_L| > T_\alpha$ and $|t_U| > T_\alpha$ for $0.3 \le \alpha \le 1$ implies the null hypothesis $\widetilde{H}_0$ is rejected and we accept the alternative hypothesis $\widetilde{H}_A$. Therefore, the two kinds of engine oils for automobiles on consumption of fuel are not the same at 5% level of significance.

Remark

The obtained result from the above test procedure in Example-2 differs by the lower value of α by 0.3 when compared with the result in Baloui Jamkhaneh and Nadi Gara [10] which is $0 \le \alpha \le 1$ when performing this test procedure using trapezoidal fuzzy interval data.

VI. WANG'S CENTROID POINT AND RANKING METHOD

Wang et al. [37] found that the centroid formulae proposed by Cheng are incorrect and have led to some misapplications such as by Chu and Tsao. They presented the correct method for centroid formulae for a generalized fuzzy number $\widetilde{A}=(a,b,c,d;w)$ as

$$\left(\bar{x}_0,\bar{y}_0\right)=\left[\frac{1}{3}\left((a+b+c+d)-\left(\frac{dc-ab}{(d+c)-(a+b)}\right)\right),\left(\frac{w}{3}\right)\left(1+\left(\frac{c-b}{(d+c)-(a+b)}\right)\right)\right]---(6.1)$$

And the ranking function associated with $\widetilde{A}$ is $R\left(\widetilde{A}\right)=\sqrt{\bar{x}_0^2+\bar{y}_0^2}---(6.2)$

For a normalized tfn, we put w = 1 in equations (6.1) so we have,

$$\left(\bar{x}_0,\bar{y}_0\right)=\left[\frac{1}{3}\left((a+b+c+d)-\left(\frac{dc-ab}{(d+c)-(a+b)}\right)\right),\left(\frac{1}{3}\right)\left(1+\left(\frac{c-b}{(d+c)-(a+b)}\right)\right)\right]---(6.3)$$

And the ranking function associated with $\widetilde{A}$ is $R\left(\widetilde{A}\right)=\sqrt{\bar{x}_0^2+\bar{y}_0^2}---(6.4)$.

Let $\widetilde{A}_i$ and $\widetilde{A}_j$ be two fuzzy numbers, (i) $R\left(\widetilde{A}_i\right)>R\left(\widetilde{A}_j\right)$ then $\widetilde{A}_i>\widetilde{A}_j$ (ii) $R\left(\widetilde{A}_i\right)>R\left(\widetilde{A}_j\right)$

then $\widetilde{A}_i > \widetilde{A}_j$ and (iii) $R(\widetilde{A}_i) = R(\widetilde{A}_j)$ then $\widetilde{A}_i = \widetilde{A}_j$.

Two-sample t-test using Wang's centroid point and ranking function

Example 6.1. Let we consider example 2, using the above relations (6.3) and (6.4), we obtain the ranks of tfns. which are tabulated below:

$R_{\widetilde{A}}(x_i)$; i=1, 2, ..., 5	$R_{\widetilde{B}}(y_j)$; j=1, 2, ..., 6
4.9163	6.6062
4.8097	4.9163
5.5766	7.2204
5.9738	6.1131
3.9205	6.9116
--	7.7196

Here, the mean values are $R_{\widetilde{A}}(\bar{x}) = 5.0394$, $R_{\widetilde{B}}(\bar{y}) = 6.5812$ and the variance values are $s_x^2 = 0.6204$, $s_y^2 = 0.9611$ and the calculated value of 't' is t = -2.8919. The tabulated value of 't' at 5% level of significance with 9 degrees of freedom is $T_\alpha = 2.262$. And $|t| > T_\alpha$ (two tailed test). $\Rightarrow$ **The null hypothesis $\widetilde{H}_0$ is rejected. Therefore, the two kinds of engine oils for automobiles on consumption of fuel are not the same at 5% level of significance.**

VII. REZVANI'S RANKING FUNCTION OF TFNS.

The centroid of a trapezoid is considered as the balancing point of the trapezoid. Divide the trapezoid into three plane figures. These three plane figures are a triangle (APB), a rectangle (BPQC) and a triangle (CQD) respectively. Let the centroids of the three plane figures be G_1, G_2 and G_3 respectively. The incenter of these centroids G_1, G_2 and G_3 is taken as the point of reference to define the ranking of generalized trapezoidal fuzzy numbers. The reason for selecting this point as a point of reference is that each centroid point are ***balancing points*** of each individual plane figure and the incenter of these centroid points is much more balancing point for a generalized trapezoidal fuzzy number. Therefore, this point would be a better reference point than the centroid point of the trapezoid.

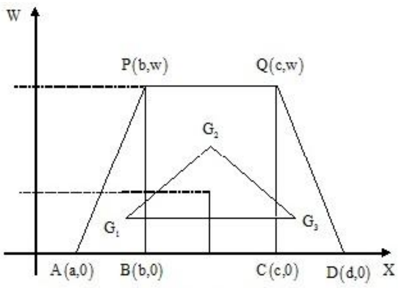

Fig.1 Centroid of centroids

Consider a generalized trapezoidal fuzzy number $\widetilde{A} = (a, b, c, d; w)$. The centroids of the three plane figures are:

$$G_1 = \left(\frac{a+2b}{3}, \frac{w}{3}\right),\ G_2 = \left(\frac{b+c}{2}, \frac{w}{2}\right) \text{ and } G_3 = \left(\frac{2c+d}{3}, \frac{w}{3}\right) --- (7.1)$$

Equation of the line G_1G_3 is $y = \frac{w}{3}$ and G_2 does not lie on the line G_1G_3. Therefore, G_1, G_2 and G_3 are non-collinear and they form a triangle. We define the incenter $I(\bar{x}_0, \bar{y}_0)$ of the triangle with vertices G_1, G_2 and G_3 of the generalized fuzzy number $\widetilde{A} = (a, b, c, d; w)$ as [30]

$$I_{\tilde{A}}\left(\overline{x}_0, \overline{y}_0\right) = \left[\frac{\alpha\left(\frac{a+2b}{3}\right)+\beta\left(\frac{b+c}{2}\right)+\gamma\left(\frac{2c+d}{3}\right)}{\alpha+\beta+\gamma}, \frac{\alpha\left(\frac{w}{3}\right)+\beta\left(\frac{w}{2}\right)+\gamma\left(\frac{w}{3}\right)}{\alpha+\beta+\gamma}\right] --- (7.2)$$

$$\text{where } \alpha = \frac{\sqrt{(c - 3b + 2d)^2 + w^2}}{6}, \beta = \frac{\sqrt{(2c + d - a - 2b)^2}}{3}, \gamma = \frac{\sqrt{(3c - 2a - b)^2 + w^2}}{6}$$

And ranking function of the trapezoidal fuzzy number $\tilde{A}=(a, b, c, d; w)$ which maps the set of all fuzzy numbers to a set of all real numbers $\left[\text{i.e. } R:[\tilde{A}] \rightarrow \mathbb{R}\right]$ is defined as $R\left(\tilde{A}\right) = \sqrt{\overline{x}_0^2 + \overline{y}_0^2} --- (7.3)$ which is the Euclidean distance from the incenter of the centroids. For a normalized tfn. we put w = 1 in equations (7.1), (7.2) and (7.3) so we have,

$$G_1 = \left(\frac{a+2b}{3}, \frac{1}{3}\right),\ G_2 = \left(\frac{b+c}{2}, \frac{1}{2}\right) \text{ and } G_3 = \left(\frac{2c+d}{3}, \frac{1}{3}\right) --- (7.4)$$

$$I_{\tilde{A}}\left(\overline{x}_0, \overline{y}_0\right) = \left[\frac{\alpha\left(\frac{a+2b}{3}\right)+\beta\left(\frac{b+c}{2}\right)+\gamma\left(\frac{2c+d}{3}\right)}{\alpha+\beta+\gamma}, \frac{\alpha\left(\frac{1}{3}\right)+\beta\left(\frac{1}{2}\right)+\gamma\left(\frac{1}{3}\right)}{\alpha+\beta+\gamma}\right] --- (7.5)$$

$$\text{where } \alpha = \frac{\sqrt{(c - 3b + 2d)^2 + 1}}{6}, \beta = \frac{\sqrt{(2c + d - a - 2b)^2}}{3} \text{ and } \gamma = \frac{\sqrt{(3c - 2a - b)^2 + 1}}{6}$$

And ranking function of the trapezoidal fuzzy number $\tilde{A}=(a, b, c, d; 1)$ is defined as $R\left(\tilde{A}\right) = \sqrt{\overline{x}_0^2 + \overline{y}_0^2} --- (7.6)$.

Two-sample t-test using Rezvani's ranking function

We now analyse the one-sample t-test by assigning rank for each normalized trapezoidal fuzzy numbers and based on the ranking grades the decisions are observed.

Example 7.1. Let we consider example 2, using the above relations (7.4), (7.5) and (7.6), we obtain the ranks of tfns. which are tabulated below:

$R_{\tilde{A}}(x_i)$; i=1, 2, ..., 5	$R_{\tilde{B}}(y_j)$; j=1, 2, ..., 6
4.7724	6.7603
4.5225	4.7724
5.6571	7.5082
5.9203	6.2594
3.7772	6.7671
--	8.0074

Here, the mean values are $R_{\tilde{A}}\left(\overline{x}\right) = 4.9299$, $R_{\tilde{B}}\left(\overline{y}\right) = 6.6791$ and the variance values are $s_x^2 = 0.7573$, $s_y^2 = 1.2556$ and the calculated value of 't' is t = -2.9125. The tabulated value of 't' at 5% level of significance with 9 degrees of freedom is $T_\alpha = 2.262$. And $|t| > T_\alpha$ (two tailed test). $\Rightarrow$ **The null hypothesis $\tilde{H}_0$ is rejected. Therefore, the two kinds of engine oils for automobiles on consumption of fuel are not the same at 5% level of significance.**

VIII. GRADED MEAN INTEGRATION REPRESENTATION (GMIR)

Let $\widetilde{A}=(a, b, c, d; w)$ be a generalized trapezoidal fuzzy number, then the **GMIR** [29] of $\widetilde{A}$ is defined by $P(\widetilde{A})=\int_0^w h\left[\frac{L^{-1}(h)+R^{-1}(h)}{2}\right]dh \,/\, \int_0^w hdh$.

Theorem 8.1. Let $\widetilde{A}=(a, b, c, d; 1)$ be a tfn. with normal shape function, where a, b, c, d are real numbers such that $a < b \le c < d$. Then the graded mean integration representation (GMIR) of $\widetilde{A}$ is $P(\widetilde{A})=\frac{(a+d)}{2}+\frac{n}{2n+1}(b-a-d+c)$.

Proof : For a trapezoidal fuzzy number $\widetilde{A}=(a, b, c, d; 1)_n$, we have $L(x)=\left(\frac{x-a}{b-a}\right)^n$ and $R(x)=\left(\frac{d-x}{d-c}\right)^n$.

Then, $h=\left(\frac{x-a}{b-a}\right)^n \Rightarrow L^{-1}(h)=a+(b-a)h^{1/n}$;

$h=\left(\frac{d-x}{d-c}\right)^n \Rightarrow R^{-1}(h)=d-(d-c)h^{1/n}$

$$\therefore P(\widetilde{A})=\left(\frac{1}{2}\int_0^1 h\left[\left(a+(b-a)h^{1/n}\right)+\left(d-(d-c)h^{1/n}\right)\right]dh\right) \Big/ \int_0^1 hdh$$

$$=\left(\frac{1}{2}\left[\frac{(a+d)}{2}+\frac{n}{2n+1}(b-a-d+c)\right]\right) \Big/ \left(\tfrac{1}{2}\right)$$

Thus, $P(\widetilde{A})=\frac{(a+d)}{2}+\frac{n}{2n+1}(b-a-d+c)$ Hence the proof.

Result 8.1. If n =1 in the above theorem, we have $P(\widetilde{A})=\frac{a+2b+2c+d}{6}$

Two-sample t-test using GMIR of tfns.

Example 8.1. Let us consider example 2, using the result-8.1 from above theorem-8.1, we get the GMIR of each tfns. $\widetilde{x}_i$, i=1, 2, ..., 5; $\widetilde{y}_j$, j=1, 2, ..., 6 which are tabulated below:

$P_{\widetilde{A}}(x_i)$; i=1, 2, ..., 5	$P_{\widetilde{B}}(y_j)$; j=1, 2, ..., 6
4.8333	6.6667
4.6667	4.8333
5.6	7.3333
5.9333	6.1667
3.8333	6.8333
--	7.8333

Here, the mean values are $P_{\widetilde{A}}(\overline{x})=4.9733$, $P_{\widetilde{B}}(\overline{y})=6.6111$ and the variance values are $s_x^2=0.6819, s_y^2=1.0852$ and the calculated value of 't' is t = -2.9078. The tabulated value of 't' at 5% level of significance with 9 degrees of freedom is $T_\alpha=2.262$. And $|t|>T_\alpha$ (two tailed test). $\Rightarrow$ **The null hypothesis $\widetilde{H}_0$ is rejected. Therefore, the two kinds of engine oils for automobiles on consumption of fuel are not the same at 5% level of significance.**

IX. TWO-SAMPLE t-TEST USING TOTAL INTEGRAL VALUE (TIV) OF TFNS.

The membership grades for a normalized tfn. $\widetilde{A}=(a, b, c, d; 1)$ is calculated by the relation [23] $\int_{Supp(\widetilde{A})} \mu_{\widetilde{A}}(x)dx = \int_a^b \left(\frac{x - a}{b - a}\right)dx + \int_b^c dx + \int_c^d \left(\frac{x - d}{c - d}\right)dx ---(9.1)$

Example 9.1. Let us consider example 2, the total integral value of first entry $\widetilde{A}_1 = (4,\ 4.5, 5, 6)$ will be

$$\int_{Supp(\widetilde{A}_1)} \mu_{\widetilde{A}_1}(x)dx = \int_4^{4.5} \left(\frac{x - 4}{0.5}\right)dx + \int_{4.5}^{5} dx + \int_5^6 \left(\frac{x - 6}{-1}\right)dx = 1.25 = I$$

The total integral values of remaining entries can be calculated in similar way, which have been tabulated below:

$\int_{Supp(\widetilde{A})} \mu_{\widetilde{A}}(x_i)dx = I;\ i = 1, 2, ..., 5$	$\int_{Supp(\widetilde{B})} \mu_{\widetilde{B}}(y_j)dy = I;\ j = 1, 2, ..., 6$
1.25	1.75
2	1.25
0.65	2
0.6	1.25
1.25	1.25
--	2

Here, the mean values are $I_{\widetilde{A}}(\overline{x}) = 1.15$, $I_{\widetilde{B}}(\overline{y}) = 1.5833$ and the variance values are $s_x^2 = 0.32375,\ s_y^2 = 0.141667$ and the calculated value of 't' is t = -1.4578. The tabulated value of 't' at 5% level of significance with 9 degrees of freedom is $T_\alpha = 2.262$. And $|t| < T_\alpha$ (two tailed test). $\Rightarrow$ **The null hypothesis $\widetilde{H}_0$ is accepted. Therefore, the two kinds of engine oils for automobiles on consumption of fuel do not differ significantly at 5% level of significance.**

X. LIOU AND WANG'S CENTROID POINT METHOD

Liou and Wang [25] ranked fuzzy numbers with total integral value. For a fuzzy number defined by definition (2.3), the total integral value is defined as

$I_T^\alpha(\widetilde{x}_i) = \alpha I_R(\widetilde{x}_i) + (1 - \alpha) I_L(\widetilde{x}_i) ---(10.1)$ where

$I_R(\widetilde{x}_i) = \int_{Supp(\widetilde{x}_i)} R_{\widetilde{x}_i}(x)dx = \int_c^d \left(\frac{x-d}{c-d}\right)dx ---(10.2)$ and

$I_L(\widetilde{x}_i) = \int_{Supp(\widetilde{x}_i)} L_{\widetilde{x}_i}(x)dx = \int_a^b \left(\frac{x-a}{b-a}\right)dx ---(10.3)$ are the ***right*** and ***left integral values*** respectively and $0 \le \alpha \le 1$.

(i) $\alpha \in [0,1]$ is the **index of optimism** which represents the **degree of optimism** of a decision maker. (ii) If $\alpha = 0$, then the total value of integral represents a **pessimistic decision maker's view point** which is equal to left integral value. (iii) If $\alpha = 1$, then the total integral value represents an **optimistic decision maker's view point** and is equal to the right integral value.(iv) If $\alpha = 0.5$ then the total integral value represents a **moderate decision maker's view point** and is equal to the mean of right and left integral values. For a decision maker, the larger the value of α is, the higher is the degree of optimism.

Two-sample t-test using Liou and Wang's centroid point method:

Example 10.1. Let we consider example 2, using the above equations (10.1), (10.2) and (10.3), we get the centroid point of first member as follows:

$$I_L\left(\tilde{x}_1\right)=\int_4^{4.5}\left(\frac{x-4}{0.5}\right)dx=1/4\;;\quad I_R\left(\tilde{x}_1\right)=\int_5^{6}\left(\frac{x-6}{-1}\right)dx=1/2.\ \text{Therefore } I_T^{\alpha}\left(\tilde{x}_1\right)=(1+\alpha)/4.$$

Similarly we can find $I_T^{\alpha}\left(\tilde{x}_i\right);\ \text{for } i = 2, ..., 5;\ I_T^{\alpha}\left(\tilde{y}_j\right);\ \text{for } j = 1, 2, ..., 6$ and the calculated values are tabulated below:

$I_T^{\alpha}\left(\tilde{x}_i\right);\ \text{for } i = 1, 2, ..., 5$	$I_T^{\alpha}\left(\tilde{y}_j\right);\ \text{for } j = 1, 2, ..., 6$
(1+α)/4	(3-α)/4
(1+2α)/4	(1+α)/4
(5-3α)/20	(3-2α)/4
(3+2α)/20	(2-α)/4
(1+α)/4	(1+α)/4
--	(3-2α)/4

Here, the mean values are $I_{T_{\tilde{A}}}^{\alpha}\left(\bar{x}\right)=(23+19\alpha)/100$, $I_{T_{\tilde{B}}}^{\alpha}\left(\bar{y}\right)=(13-4\alpha)/24$ and the variance values are $s_x^2=0$, $s_y^2=0$. **As the variance values vanish for both the sample data x and y, we can not decide any conclusion for this case using Liou and Wang's method.**

XI. THORANI'S RANKING METHOD

As per the description in Salim Rezvani's ranking method, Thorani et al. [31] presented a different kind of centroid point and ranking function of tfns. The incenter $I_{\tilde{A}}\left(\bar{x}_0,\ \bar{y}_0\right)$ of the triangle [Fig. 1] with vertices G_1, G_2 and G_3 of the generalized tfn. $\tilde{A}=\left(a, b, c, d; w\right)$ is given by,

$$I_{\tilde{A}}\left(\bar{x}_0,\ \bar{y}_0\right)=\left[\frac{\alpha\left(\frac{a+2b}{3}\right)+\beta\left(\frac{b+c}{2}\right)+\gamma\left(\frac{2c+d}{3}\right)}{\alpha+\beta+\gamma},\ \frac{\alpha\left(\frac{w}{3}\right)+\beta\left(\frac{w}{2}\right)+\gamma\left(\frac{w}{3}\right)}{\alpha+\beta+\gamma}\right]---(11.1)$$

$$\text{where } \alpha=\frac{\sqrt{\left(c-3b+2d\right)^2+w^2}}{6},\beta=\frac{\sqrt{\left(2c+d-a-2b\right)^2}}{3},\gamma=\frac{\sqrt{\left(3c-2a-b\right)^2+w^2}}{6}$$

And the ranking function of the generalized tfn. $\tilde{A}=\left(a, b, c, d; w\right)$ which maps the set of all fuzzy numbers to a set of real numbers is defined as $R\left(\tilde{A}\right)=x_0\times y_0---(11.2)$. For a normalized tfn., we put w = 1 in equations (11.1) and (11.2) so we have,

$$I_{\tilde{A}}\left(\bar{x}_0,\ \bar{y}_0\right)=\left[\frac{\alpha\left(\frac{a+2b}{3}\right)+\beta\left(\frac{b+c}{2}\right)+\gamma\left(\frac{2c+d}{3}\right)}{\alpha+\beta+\gamma},\ \frac{\alpha\left(\frac{1}{3}\right)+\beta\left(\frac{1}{2}\right)+\gamma\left(\frac{1}{3}\right)}{\alpha+\beta+\gamma}\right]---(11.3)$$

$$\text{where } \alpha=\frac{\sqrt{\left(c-3b+2d\right)^2+1}}{6},\beta=\frac{\sqrt{\left(2c+d-a-2b\right)^2}}{3}\ \text{and } \gamma=\frac{\sqrt{\left(3c-2a-b\right)^2+1}}{6}$$

And for $\tilde{A}=\left(a, b, c, d; 1\right)$, $R\left(\tilde{A}\right)=x_0\times y_0---(11.4)$

Two-sample t-test using Thorani's ranking method of tfns.

Example 11.1. Let us consider example 2, using the above relations (11.3) and (11.4), we get the ranks of each tfns. $\tilde{x}_i,\ i=1, 2, ..., 5;\ \tilde{y}_j,\ j=1, 2, ..., 6$ which are tabulated below:

$R_{\tilde{A}}(x_i)$; i=1, 2, ..., 5	$R_{\tilde{B}}(y_j)$; j=1, 2, ..., 6
1.9703	2.8028
1.8726	1.9703
2.3114	3.1172
2.4098	2.5883
1.5559	2.7991
--	3.3251

Here, the mean values are $R_{\tilde{A}}(\bar{x}) = 2.0240$, $R_{\tilde{B}}(\bar{y}) = 2.7671$ and the variance values are $s_x^2 = 0.1191$, $s_y^2 = 0.2206$ and the calculated value of 't' is t = -3.0191. The tabulated value of 't' at 5% level of significance with 9 degrees of freedom is $T_\alpha = 2.262$. And $|t| > T_\alpha$ (two tailed test). $\Rightarrow$ **The null hypothesis $\tilde{H}_0$ is rejected. Therefore, the two kinds of engine oils for automobiles on consumption of fuel differ significantly at 5% level of significance.**

XII. GENERAL CONCLUSION

The decisions obtained from various methods are tabulated below for the acceptance of null hypothesis.

Acceptance of null hypothesis $\tilde{H}_0$ at 5% level of significance						
α-cut	**Wang**	**Rezvani**	**GMIR**	**TIV**	**L&W**	**Thorani**
×	×	×	×	✓	Cannot be decided	×

Observing the decisions obtained from α-cut interval method, for example-2, the null hypothesis is rejected for the level of $0.3 \le \alpha \le 1$ at both l.l.m. and u.l.m. On the other hand, using Liou & Wang's method (L&W), we cannot conclude any decision as the variance values are zero. Moreover, the method of total integral value of tfns. (TIV) do not provide reliable results as it accepts the null hypothesis $\tilde{H}_0$ while other methods are rejecting $\tilde{H}_0$ and the decisions obtained from Wang's ranking method, Rezvani's ranking method, GMIR and Thorani's ranking method of tfns. provide parallel discussions.

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