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Oscillation results for second order nonlinear neutral delay dynamic equations on time scales

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Abstract: *In this paper, we establish sufficient conditions for the oscillation of solutions of second order neutral delay dynamic equations*

$$[r(t)(x(t)+p(t)x(\tau(t)))^{\Delta}]^{\Delta}+q(t)f(x(\delta(t)))=0$$

on an arbitrary time scale $\mathbb{T}$.

Keywords: *Oscillation; Neutral; Dynamic equations; Time scale.*

I. Introduction

The theory of time scales was introduced by Hilger in his Ph.D. thesis [1] in 1988 in order to unify continuous and discrete analysis. Since then, many authors have considered the time scales theory and its usage in various aspects of applied mathematics, see [2-9] and the references cited therein. In particular, we refer to the books of Bohner and Peterson [3, 4] as detailed references for the time scales calculus.

Using the theory of time scales helps to avoid proving results twice, once for differential equations and once for difference equations [3, 4]. A time scale $\mathbb{T}$ is a non empty closed subset of real numbers. In this theory, the so called *dynamic equations* unify the classical theories for differential and difference equations if this time scale is equal to the reals and to the integers, respevtively. Moreover, the new theory of dynamic equations is able to extend these classical cases to cases 'in between', for example to the so-called *q-difference equations*, see [3, 4].

During the last few years, there has been a growing interest in obtaining oscillation criteria for the neutral dynamic equations on time scales, see [5-9] and the references cited therein.

This paper is concerned with the oscillation of the second order neutral delay dynamic equations of the form

$$[r(t)(x(t)+p(t)x(\tau(t)))^{\Delta}]^{\Delta}+q(t)f(x(\delta(t)))=0,\quad t\in[t_0,\infty)_{\mathbb{T}} \tag{1}$$

Where $\mathbb{T}$ is an arbitrary time scale unbounded from above, and $[t_0,\infty)_{\mathbb{T}}:=[t_0,\infty)\cap\mathbb{T}$. Throughout the paper it is assumed that:

(A_1) $r\in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ with $r(t)>0$;

(A_2) $p\in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ and $, 0\le p(t)\le p_0<1$;

(A_3) $q\in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ and $q(t)>0$;

(A_4) $\delta\in C^1_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{T})$ with $\delta(t)\le t$, $\lim_{t\to\infty}\delta(t)=\infty$;

(A_5) $\tau\in C^1_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{T})$ with $\tau(t)\le t$, $\lim_{t\to\infty}\tau(t)=\infty$;

(A_6) f is continuous function with $\frac{f(x)}{x}\ge K>0$ for $x\ne 0$, where K is a constant.

We construct sufficient conditions for the oscillation of equation (1) for both cases

$$\int_t^{\infty} r^{-1}(s)\Delta s=\infty, \tag{2}$$

and

$$\int_t^{\infty} r^{-1}(s)\Delta s<\infty. \tag{3}$$

By a solution of (1) we mean a real-valued function x which satisfies (1) and $\sup\{|x(t)|: t \geq t_x\} > 0$ for any $t_x \geq t_0$. Such a solution is said to be oscillatory if it has no constant sign eventually, and nonoscillatory otherwise.

There are numerous numbers of oscillation criteria obtained for oscillation for neutral differential equations, i.e. when $\mathbb{T}=\mathbb{R}$. These results have considered many different forms of equations with different conditions. Relatively, few oscillation results for dyanmic equations on an arbitrary time scale $\mathbb{T}$ are known. Moreover, considering more general forms of nonlinear dyanmic equations on an arbitrary time scale $\mathbb{T}$ is still a challenge.

Several authors have considered the oscillatory behaviour of (1) and related forms. In particular, Agarwal et al. [2] obtained sufficient conditions for the oscillation of the second ordered nonlinear neutral delay dynamic equations

$$[r(t)[(x(t) + p(t)x(t-\tau)^{\Delta}]^{\alpha}]^{\Delta} + f(t, x(t-\delta)) = 0,$$

where $\alpha > 0$ is a quotient of odd positive integers, τ and δ are positive constants, $r(t)$ and $p(t)$ are real-valued positive functions defined on $\mathbb{T}$ and the condition (2) holds. Saker [7] also provided an oscillation criteria for this equation considering the same assumption (2).

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$$[r(t)[(x(t) + p(t)x(t-\tau))^{\Delta}]^{\alpha}]^{\Delta} + q(t)x^{\alpha}(t-\delta)) = 0$$

were presented in Li et al. [6] considering the condition (2). Where $\alpha > 0$ is a quotient of odd positive integers with $r(t)$ and $p(t)$ are real-valued positive functions defined on $\mathbb{T}$.

Thandapan and Piramanantham [8] established an extended oscillation criteria for the dynamic equations

$$[r(t)[(x(t) + p(t)x(t-\tau))^{\Delta}]^{\alpha}]^{\Delta} + q(t)x^{\beta}(t-\delta)) = 0,$$

where $\alpha \geq 1$, and $\beta > 0$ are quotients of odd positive integers, τ and δ are positive constants. Both conditions (2) and (3) were considered.

Zhang et al. [9] considered the oscillatory behaviour of delay dynamic equations

$$[r(t)(x(t) + p(t)x(\tau(t)))^{\Delta}]^{\Delta} + q(t)x(\delta(t)) = 0,$$

under the assumption (2).

Yang et al. [10] and Zhang et al. [11] discussed oscillation of the second-order nonlinear neutral dynamic equations with distributed deviating arguments.

The aim of this paper is to obtain new sufficient conditions for equation (1) to oscilate. Both cases (2) and (3) are considered. One of our results includs a result of Al-Hamouri and Zein [12] when $\mathbb{T}=\mathbb{R}$.

In this paper, we assume that the reader is familiar with the time scale calculus. For further reading we refer the reader to [3,4]. To prove our results, we will use the following lemma which is one form of chain rules on time scales.

Lemma 1.1 *[[3], Theorem 1.93] Assume that* $g: \mathbb{T} \to \mathbb{R}$ *is strictly increasing,* $\tilde{\mathbb{T}} = g(\mathbb{T})$ *is a time scale and* $f: \tilde{\mathbb{T}} \to \mathbb{R}$*.. If* $f^{\tilde{\Delta}}(g(t))$ *and* $g^{\Delta}(t)$ *exist for* $t \in \tilde{\mathbb{T}}$ *then*

$$(f(g(t)))^{\Delta} = f^{\tilde{\Delta}}(g(t))\, g^{\Delta}(t). \tag{4}$$

II. The Main Result

Theorem 2.1 *Let (2) holds, and suppose that* $\delta^{\Delta}(t) > 0$. *If*

$$\int^{\infty} q(t)\Delta t = \infty, \tag{5}$$

then every solution of (1) is oscillatory .

Proof. Suppose to the contrary that $x(t)$ is a nonoscillatory solution of equation (1). Without loss of generality, we may assume that $x(t)$ is eventually positive (the proof is similar when $x(t)$ is eventually negative). That is, let $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\delta(t)) > 0$ for $t \geq t_1 \geq t_0$.

Set

$$z(t) = x(t) + p(t)x(\tau(t)) \tag{6}$$

By (A_6) we have

$$f(x(\delta(t))) \geq K(x(\delta(t))) > 0. \tag{7}$$

Using (7) together with (6) and (1) we get

$$(r(t)z^{\Delta}(t))^{\Delta} = -q(t)f(x(\delta(t))) \leq -Kq(t)x(\delta(t)) < 0.$$

Thus $r(t)z^{\Delta}(t)$ is an eventually decreasing function. Since $r(t)$ is positive $z^{\Delta}(t)$ is either negative or positive eventuality. If $z^{\Delta}(t) < 0$ for $t \geq t_2 \geq t_1$, then by using the fact that $r(t)z^{\Delta}(t)$ is decreasing, we have

$$z^{\Delta}(t) \leq \frac{r(t_2)}{r(t)} z^{\Delta}(t_2), \quad \text{for } t \geq t_2 .$$

Integrating this inequality and using (2), we get

$$\lim_{t\to\infty} z(t) \leq r(t_2)z^{\Delta}(t_2)\int_{t_2}^{\infty} \frac{\Delta t}{r(t)} - z(t_2) = -\infty .$$

This contradicts the fact that $z(t)$ is eventually positive. Hence, $z^{\Delta}(t) > 0$ eventually, say that $z(\delta(t)) > 0,\ z^{\Delta}(t) > 0$ and $z^{\Delta}(\delta(t)) > 0$ for $t \geq t_2 \geq t_1$. In this case $z(t)$ is increasing. Using this fact with (6) and (A_2) we get

$$\begin{aligned} x(t) &= z(t) - p(t)x(\tau(t)) \\ &\geq z(t) - p(t)z(\tau(t)) \\ &\geq z(t)(1 - p(t)) \\ &\geq z(t)(1 - p_0). \end{aligned}$$

Then using the last inequality together with (A_6) we obtain

$$f(x(\delta(t))) \geq K(1 - p_0)z(\delta(t)), \quad \text{for } t \geq t_2. \tag{8}$$

Now consider the term $(z(\delta(t)))^{\Delta}$, it can be differentiated by the chain rule (4) as

$$(z(\delta(t)))^{\Delta} = z^{\Delta}(\delta(t))\delta^{\Delta}(t) > 0 \tag{9}$$

Define

$$w(t) = r(t)z^{\Delta}(t)\frac{1}{z(\delta(t))}, \tag{10}$$

then $w(t) > 0$ for $t \geq t_2$.

By differentiating $w(t)$, taking into consideration (9) and the fact that $w(t) > 0$, we have the following

$$\begin{aligned} (w(t))^{\Delta} &= (r(t)z^{\Delta}(t))^{\Delta}\frac{1}{z(\delta(t))} - \frac{(z(\delta(t)))^{\Delta}}{z(\delta(\sigma(t)))\,z(\delta(t))} r(\sigma(t))\,z^{\Delta}(\sigma(t)) \\ &= (r(t)z^{\Delta}(t))^{\Delta}\frac{1}{z(\delta(t))} - \frac{(z(\delta(t)))^{\Delta}}{z(\delta(t))} w(\sigma(t)) \\ &\leq (r(t)z^{\Delta}(t))^{\Delta}\frac{1}{z(\delta(t))} = -q(t)f(x(\delta(t))\frac{1}{z(\delta(t))}. \end{aligned}$$

Using (8) in the last inequality we get that

$$w^{\Delta}(t) \le -K(1-p_0)q(t). \tag{11}$$

Integrating (11) and using the condition (5), we obtain

$$\lim_{t\to\infty} w(t) \le -K(1-p_0)\int_{t_2}^{\infty} q(t)\Delta t + w(t_2) = -\infty .$$

This contradicts the fact that $w(t)$ is positive and this completes the proof. □

In the next results besides conditions (A_1) - (A_6) we further assume that:

(A_7) The deviating arguments are commute, i.e. $\delta \circ \tau = \tau \circ \delta$.

Theorem 2.2 *Let (3) holds, and there exists a positive constant* τ_0 *with* $0 < \tau_0 \le \tau^{\Delta}(t)$. *If*

$$\int^{\infty} \left[\frac{1}{r(t)} \int Q(s)\Delta s \right] \Delta t = \infty , \tag{12}$$

where

$$Q(t) = min\ \{q(t), q(\tau(t))\} ,$$

then every solution of (1) is oscillatory or tends to zero eventually .

Proof. Assume that $x(t)$ is a nonoscillatory solution of equation (1). Without loss of generality, assume that $x(t)$ is eventually positive. That is, let $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\delta(t)) > 0$ for $t \ge t_1 \ge t_0$. Set $z(t)$ as in (6), also by (A_6) we obtain (7).
Let

$$u(t) = r(t) z^{\Delta}(t), \tag{13}$$

then

$$(u(t))^{\Delta} = -q(t)\, f(x(\delta(t))) \le -Kq(t) x(\delta(t)) < 0. \tag{14}$$

Thus $u(t)$ is strictly decreasing and so with constant sign eventually. Since $r(t)$ is positive $z^{\Delta}(t)$ is eventuality of one sign. Hence, either

$$z^{\Delta}(t) > 0, \quad \text{for } t \ge t_2 \ge t_1, \tag{15}$$

or

$$z^{\Delta}(t) < 0, \quad \text{for } t \ge t_2 \ge t_1. \tag{16}$$

If (15) holds, then equation (1) implies that

$$u^{\Delta}(\tau(t)) + q(\tau(t))\ f(x(\delta(\tau(t)))) = 0.$$

Using the chain rule (4), also taking into consideration that $\tau \circ \delta = \delta \circ \tau$ together with (A_5) and (A_6) we obtain

$$\frac{p_0}{\tau_0}[u(\tau(t))]^{\Delta} + Kp_0 q(\tau(t))\, x(\tau(\delta(t))) \le 0.$$

Adding this inequality to (14) we get

$$(u(t))^{\Delta} + Kq(t) x(\delta(t)) + \frac{p_0}{\tau_0}[u(\tau(t))]^{\Delta} + Kp_0 q(\tau(t))\, x(\tau(\delta(t))) \le 0,$$

this leads to

$$(u(t))^{\Delta} + KQ(t)[x(\delta(t)) + p_0 x(\tau(\delta(t)))] + \frac{p_0}{\tau_0}[u(\tau(t))]^{\Delta} \le 0. \tag{17}$$

By (6) we get that $z(t) \le x(t) + p_0 x(\tau(t))$ and this with (17) lead to

$$(u(t))^{\Delta} + KQ(t)z(\delta(t)) + \frac{p_0}{\tau_0}[u(\tau(t))]^{\Delta} \le 0. \tag{18}$$

Using the fact that $z(t)$ is increasing, one can find $t_3 \ge t_2$ such that

$$z(\delta(t)) \ge z(t_3) = c \text{ , for all } t \ge t_3 ,$$

where c is a positive constant, then from (18) we get

$$Q(t) \le \left(-(u(t))^{\Delta} - \frac{p_0}{\tau_0}[u(\tau(t))]^{\Delta} \right) \frac{1}{Kc}. \tag{19}$$

Integrating this inequality from t_3 *to* t , we have

$$\int_{t_3}^{t} Q(t)\Delta t \le \left(-u(t) + u(t_3) + \frac{p_0}{\tau_0}(-u(\tau(t)) + u(\tau(t_3))) \right) \frac{1}{Kc}.$$

Since $u(t)$ is positive, we get

$$\int_{t_3}^{t} Q(s)\Delta s \le \left(u(t_3) + \frac{p_0}{\tau_0} u(\tau(t_3)) \right) \frac{1}{Kc} = C_0. \tag{20}$$

Note that C_0 is a positive constant. Dividing by $r(t)$ and integrating, we obtain

$$\int_{t_3}^{\infty} \frac{1}{r(t)} \int_{t_3}^{t} Q(s)\Delta s \Delta t \le C_0 \int_{t_3}^{\infty} \frac{\Delta t}{r(t)} < \infty .$$

Which contradicts the condition (12).

Suppose that (16) holds, i.e. $z^{\Delta}(t) < 0,$ for $t \ge t_2$, then $z(t)$ is decreasing. Since $z(t) > 0$ we have

$$\lim_{t \to \infty} z(t) = L \ge 0.$$

Now, recall the definition of $z(t)$ then

$$\begin{aligned} x(t) &= z(t) - p(t)x(\tau(t)) \\ &\ge z(t) - p(t)z(\tau(t)) \\ &\ge z(\tau(t))\left(\frac{z(t)}{z(\tau(t))} - p_0 \right). \end{aligned} \tag{21}$$

Let $L > 0$. Following [13], since $z(t)$ is decreasing, for every $\varepsilon > 0$ there exists $t_3 \ge t_2$ such that

$$L \le z(t) \le z(\tau(t)) \le L + \varepsilon$$

for all $t \ge t_3$. From this we can conclude that

$$\frac{z(t)}{z(\tau(t))} \ge \frac{L}{L + \varepsilon}, \text{ for } t \ge t_3 .$$

Using this in (21) we get

$$x(t) \ge z(\tau(t))\left(\frac{L}{L + \varepsilon} - p_0 \right). \tag{22}$$

Choose $\varepsilon > 0$ such that $\frac{L}{L+\varepsilon} - p_0 = L_0 > 0$ then

$$x(t) \geq z(\tau(t)) L_0, \text{ for } t \geq t_3. \tag{23}$$

Substituting (23) in (14) we have

$$(u(t))^{\Delta} + KL_0 Q(t) z(\tau(\delta(t))) \leq (u(t))^{\Delta} + KL_0 q(t) z(\tau(\delta(t))) \leq 0. \tag{24}$$

For $t \geq t_4 \geq t_3$, $z(\tau(\delta(t))) \geq L$, then (24) becomes

$$(r(t) z^{\Delta}(t))^{\Delta} \leq -KL_0 LQ(t). \tag{25}$$

Integrating this inequality from t_4 to t we get

$$r(t) z^{\Delta}(t) \leq -KL_0 L \int_{t_4}^{t} Q(s)\Delta s + r(t_4) z^{\Delta}(t_4).$$

Hence

$$z^{\Delta}(t) \leq -KL_0 L \frac{1}{r(t)} \int_{t_4}^{t} Q(s)\Delta s.$$

Integrating again we get

$$z(t) \leq -K L_0 L \int_{t_4}^{t} \frac{1}{r(v)} \int_{t_4}^{v} Q(s)\Delta s \Delta v + z(t_4).$$

Taking limit as $t \to \infty$ and using (12), we get that $\lim_{t\to\infty} z(t) = -\infty$. This contradicts the fact that $z(t) > 0$.

If $L = 0$, i.e $\lim_{t\to\infty} z(t) = 0$. Since $0 < x(t) \leq z(t)$ taking the limit as $t \to \infty$ implies that $x(t) \to 0$ eventually and this completes the proof. □

Theorem 2.3 *Let (3) holds and there exists a positive constant* τ_0 *with* $0 < \tau_0 \leq \tau^{\Delta}(t)$. *If*

$$\int^{\infty} Q(t)\Delta t = \infty, \tag{26}$$

and there exists a positive function $\eta(t)$ *with* $\eta^{\Delta}(t) > 0$ *such that*

$$\int^{\infty} \left[\frac{1}{r(t)\eta(t)} \int^{t} \eta(s) Q(s)\Delta s \right] \Delta t = \infty, \tag{27}$$

where $Q(t)$ *is defined as in Theorem 2.2, then every solution of (1) is oscillatory or tends to zero eventually* .

Proof. Assume that equation (1) has a non-oscillatory solution $x(t)$. Without loss of generality, we assume that there exists a $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\delta(t)) > 0$ for all $t \geq t_1$.

Proceeding as in the proof of Theorem 2.2, we conclude that $r(t) z^{\Delta}(t)$ is decreasing and $z^{\Delta}(t)$ is eventually of one sign. Hence, either (15) or (16) holds.

If (15) holds, by proceeding as in the proof of Theorem 2.2 we obtain (20) which reads

$$\int_{t_3}^{t} Q(s)\Delta s \le \left(u(t_3) + \frac{p_0}{\tau_0} u(\tau(t_3)) \right) \frac{1}{Kc} = C_0 .$$

Taking the limit as $t \to \infty$, we have $\int_{t_3}^{\infty} Q(s)\Delta s < \infty$, and this contradicts with (26).

If (16) holds, proceed as in the proof of Theorem 2.2 to get (25), which reads

$$(r(t)z^{\Delta}(t))^{\Delta} \le -KL_0 LQ(t), \quad for \quad t \ge t_4 \tag{28}$$

Define

$$w(t) = \eta(t)u(t), \tag{29}$$

then by using the fact that $u^{\sigma}(t)$ is negative we obtain

$$w^{\Delta}(t) = \eta^{\Delta}(t)u^{\sigma}(t) + \eta(t)(u(t))^{\Delta}$$
$$< \eta(t)(u(t))^{\Delta}.$$

Using (28) with this inequality, we get

$$w^{\Delta}(t) < -KL_0 L\,\eta(t)Q(t). \tag{30}$$

Integrating inequality (30) from t_4 to t we get

$$w(t) < w(t_4) - KL_0 L\int_{t_4}^{t} \eta(s)Q(s)\Delta s.$$

By using the definitions of $w(t)$ and $u(t)$, with the fact that $w(t) < 0$, we obtain

$$\eta(t)r(t)z^{\Delta}(t) < -KL_0 L\int_{t_4}^{t} \eta(s)Q(s)\Delta s,$$

or

$$z^{\Delta}(t) \le \frac{-KL_0 L}{\eta(t)r(t)}\int_{t_4}^{t} \eta(s)Q(s)\Delta s. \tag{31}$$

Integrating (31) we get

$$z(t) \le -KL_0 L\int_{t_4}^{t} \frac{1}{\eta(u)r(u)} \left[\int_{t_4}^{u} \eta(s)Q(s)\Delta s \right] \Delta u - z(t_4)$$

Taking the limit as $t \to \infty$ and by using (27), a contradiction with $z(t) > 0$ is obtained.

If $L = 0$, i.e $\lim_{t \to \infty} z(t) = 0$. Since $0 < x(t) \le z(t)$ taking the limit as $t \to \infty$ implies that $x(t) \to 0$ eventually and this completes the proof. □

References

[1] S. Hilger, *Ein Maßkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten*,Ph.D. thesis, Universität Würzburg, 1988.

[2] R. Agarwal , D. O'Regan, and S. Saker, *Oscillation criteria for second-order nonlinear neutral delay dynamic equations*, Journal of Mathematical Analysis and Applications, vol. 300, no. 1, pp. 203-217, 2004.

[3] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*,Birkhäuser Boston, Massachusetts, 2001.

[4] M. Bohner and A. Peterson, *Advances in Dynamic Equations on Time Scales*, Birkhäuser Boston, Massachusetts, 2003.

[5] C. Gao , T. Li, S. Tang, and E. Thandapani, *Oscillation theorems for second-Order neutral functional dynamic equations on time scales*, Electronic Journal of differential Equations, vol. 2011, no. 101, 2011.

[6] T. Li , R. Agarwal, and M. Bohner, *Some oscillation results for second-order neutral dynamic equations*, Hacettepe Journal of Mathematics and Statistics, vol. 41, no. 5, pp. 715-721, 2012.

[7] S. Saker, *Oscillation of second-order nonlinear neutral delay dynamic equations on time scales*, Journal of of Computational and Apllied Mathematics, vol. 187, no. 2, pp. 123-141, 2006.

[8] E. Thandapani and V. Piramanantham, *Oscillation criteria for second order nonlinear neutral dynamic equations on time scales*, Tamkang Journal of Mathematics, vol. 43, pp. 109-122, 2012.

[9] C. Zhang , R. Agarwal, M. Bohner, and T. Li, *New oscillation results for second-order neutral delay dynamic equations*, Advances in difference equations, vol. 2012, 2012.

[10] Q. Yang, B. Jia, and Y. Xu, *Nonlinear oscillation of second-order neutral dynamic equations with distributed delay*,Mathematical Methods in the Applied Sciences ,vol. 39, pp. 202–213, 2016.

[11] C. Zhang, R. Agarwal, M. Bohner, T. Li, *Oscillation of second-order nonlinear neutral dynamic equations with noncanonical operators*, Bulletin of the Malaysian Mathematical Sciences Society, vol. 38, pp. 761-778, 2015.

[12] R. Al-Hamouri and A. Zein, *Oscillation criteria for certain even order neutral delay differential equations*, International Journal of Differential Equations, vol. 2014, Article ID 437278, 2014.

[13] R. Al-Hamouri and A. Zein, *Oscillation results of higher order nonlinear neutral delay differential equations*, Electronic Journal of Qualitative Theory of Differential Equations, 2014, No. 19, 1-7, 2014.