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On a Generalized $\beta R -$ Birecurrent Affinely Connected Space

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Abstract: *In the present paper, we introduced a Finsler space whose Cartan's third curvature tensor R^i_{jkh} satisfies the generalized βR −birecurrence property which posses the properties of affinely connected space will be characterized by*

$$\mathcal{B}_m \mathcal{B}_n R^i_{jkh} = a_{mn} R^i_{jkh} + b_{mn} \left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right),$$

where a_{mn} and b_{mn} are non-zero covariant tensors field of second order called recurrence tensors field, such space is called as a generalized βR −birecurrent affinely connected space. Ricci tensors H_{jk} and R_{jk}, the curvature vector H_k and the curvature scalars H and R of such space are non-vanishing. Some conditions have been pointed out which reduce a generalized βR −birecurrent affinely connected space $F_n (n > 2)$ into Finsler space of scalar curvature.
Keywords: *Finsler space, Generalized βR −birecurrent affinely connected space, , Finsler space of curvature scalar.*

I. Introduction

H.S. Ruse[4] considered a three dimensional Riemannian space having the recurrent of curvature tensor and he called such space as *Riemannian space of recurrent curvature*. This idea was extended to n-dimensional Riemannian and non- Riemannian space by A.G. Walker [1],Y.C. Wong [9] ,Y.C. Wong and K. Yano [10] and others . S. Dikshit [8] introduced a Finsler space whose Berwald curvature tensor H^i_{jkh} satisfies the recurrence property in sense of Berwald, F.Y.A.Qasem and A.A.M.Saleem [2] discussed general Finsler space for the hv −curvature tensor U^i_{jkh} satisfies the birecurrence property with respect to Berwald's coefficient G^i_{jk} and they called it *UBR- Finsler space.* P.N.pandey, S.Saxena and A.Goswami [6] introduced a Finsler space whose Berwald curvature tensor H^i_{jkh} satisfies generalized the recurrence property in the sense of Berwald they called such space generalized H-recurrent Finsler space .F.Y.A.Qaasem and W.H.A.Hadi [3] introduced and studied generalizedβR −birecurrentFinslerspace .

An affinely connected space or Berwald space characterized by any one of the two equivalent conditions

(1.1) a) $G^i_{jkh} = 0$ and b) $C_{ijk|h} = 0$.
Also it has the following properties
(1.2) a) $\mathcal{B}_k \, g_{ij} = 0$ and b) $\mathcal{B}_k \, g^{ij} = 0$.
Berwald covariant derivative of y^i vanishes ,i.e.
(1.3) $\mathcal{B}_k \, y^i = 0$.
The vector y_i, its associatine y^i and the metric tensor g_{ij} given by
(1.4) a) $y_i \, y^i = F^2$ and b) $g_{ij} = \dot\partial_i y_j = \dot\partial_j y_i$.
The processes of Berwald's covariant differentiation and the partial differentiation commute according to
(1.5) $(\dot\partial_k \mathcal{B}_h - \mathcal{B}_k \dot\partial_h) T^i_j = T^r_j G^i_{khr} - T^i_r G^r_{khj}$.
The tensor H^i_{jkh} satisfies the relation
(1.6) $H^i_{jkh} = \dot\partial_j H^i_{kh}$.
The $h(v) -$ torsion tensor H^i_{kh} satisfies
(1.7) a) $H^i_{kh} \, y^k = H^i_h$ and b) $R^i_{jkh} \, y^j = H^i_{kh}$.
Also we have the following relations
(1.8) a) $H_{jk} = H^i_{jki}$, b) $H_k = H^i_{ki}$ and c) $H = \frac{1}{n-1} H^i_i$.
where H_{jk} and H are called *H-Ricci tensor* [5] and *curvature scalar*, respectively. Since the contraction of the indices doesn't effect the homogeneity in y^i, hence the tensors H_{rk} , H_r and the scalar H are homogeneous of degree zero, one and two in y^i, respectively . The above tensors are also connected by
(1.9) a) $H_{jk} \, y^j = H_k$, b) $H_{jk} = \dot\partial_j H_k$ and c) $H_k \, y^k = (n-1)H$.

The necessary and sufficient condition for a Finsler space $F_n (n > 2)$ to be a Finsler space of curvature scalar is given by

(1.10) $H_h^i = F^2 R(\delta_h^i - l^i l_h)$,

where H_h^i is the deviation of the curvature tensor H^i_{jkh} .

$R -$ Ricci tensor R_{jk} of the curvature tensor R^i_{jkh} , the deviation tensor R^i_h and the curvature scalar R are given by

(1.11) a) $R^i_{jki} = R_{jk}$, b) $R^i_{jkh} g^{jk} = R^i_h$ and c) $g^{jk} R_{jk} = R$.

2.A Generalized βR −Birecurrent Affinely Connected Space

A Finsler space whose Cartan's third curvature tensor R^i_{jkh} satisfies the following generalized $\beta R -$ recurrence condition

(2.1) $\mathcal{B}_n R^i_{jkh} = \lambda_n R^i_{jkh} + \mu_n (\delta^i_k g_{jh} - \delta^i_h g_{jk})$, $R^i_{jkh} \neq 0$,

where λ_n and μ_n are non-zero covariant vectors field and called the *recurrence vectors field,* known as *a generalized βR- recurrent space*.

Differentiating (2.1) covariantly with respect to x^m in sense of Berwald and using (1.2a) ,we get

(2.2) $\mathcal{B}_m \mathcal{B}_n R^i_{jkh} = a_{mn} R^i_{jkh} + b_{mn} (\delta^i_k g_{jh} - \delta^i_h g_{jk})$,

where $a_{mn} = \mathcal{B}_m \lambda_n + \lambda_n \lambda_m$ and $b_{mn} = \lambda_n \mu_m + \mathcal{B}_m \mu_n$ are non-zero covariant tensors field of second order .

Definition2.1. A Finsler space whose Cartan's third curvature tensor R^i_{jkh} satisfies the condition (2.2) will be called *generalized βR-birecurrent affinely connected space* ,we shall denote it $G\beta R - BR -$ affinely connected space.

Let us consider a $G\beta R - BR -$ affinely connected space .

Transvecting the condition (2.2) by y^j , using (1.3) and (1.7b),we get

(2.3) $\mathcal{B}_m \mathcal{B}_n H^i_{kh} = a_{mn} H^i_{kh} + b_{mn} (\delta^i_k y_h - \delta^i_h y_k)$.

Further , transvecting (2.3) by y^k , using (1.3) , (1.8a) and (1.4a) , we get

(2.4) $\mathcal{B}_m \mathcal{B}_n H^i_h = a_{mn} H^i_h + b_{mn} (y^i y_h - \delta^i_h F^2)$.

Thus, we conclude

Theorem2.1. In $G\beta R - BR -$ affinely connected space , Berwald covariant derivative of second order for the $h(v)$ −torsion tensor H^i_{kh} and the deviation tensor H^i_h , given by the conditions (2.3) and (2.4), respectively.

Contracting the indices i and h in (2.3) and (2.4) , separately and using (1.8b) and (1.8c) , we get.

(2.5) $\mathcal{B}_m \mathcal{B}_n H_k = a_{mn} H_k + (1-n) b_{mn} y_k$

and

(2.6) $\mathcal{B}_m \mathcal{B}_n H = a_{mn} H - b_{mn} F^2$,

respectively .

The conditions (2.5) and (2.6), show that the curvature vector H_k and the curvature scalar H can't vanish, because the vanishing of any one of them would imply $b_{mn} = 0$, a contradiction .

Thus, we conclude

Theorem2.2. In $G\beta R - BR -$ affinely connected space, the curvature vector H_k and the curvature scalar H are non-vanishing .

Further, transvecting the condition (2.2) by g^{jk} , using (1.2b) and (1.11b), we get

(2.7) $\mathcal{B}_m \mathcal{B}_n R^i_h = a_{mn} R^i_h$.

Thus, we conclude

Theorem2.3. In $G\beta R - BR -$ affinely connected space, the deviation tensor R^i_h behaves as birecurrent.

Contracting the indices i and h in the condition (2.2) and using (1.11a), we get

(2.8) $\mathcal{B}_m \mathcal{B}_n R_{jk} = a_{mn} R_{jk} + (1-n) b_{mn} g_{jk}$.

Transvecting the condition (2.8) by g^{jk} , using (1.2b) and (1.11c), we get

(2.9) $\mathcal{B}_m \mathcal{B}_n R = a_{mn} R + (1-n)\, b_{mn}$.

The conditions (2.8) and (2.9), show that the R −Ricci tensor R_{jk} and the curvature scalar R can't vanish, because the vanishing of any one of them would imply $b_{mn} = 0$, a contradiction .

Thus, we conclude

Theorem 2.4. In $G\beta R - BR -$ affinely connected space, R −Ricci tensor R_{jk} and the curvature scalar R are non-vanishing.

Now, differentiating the condition (2.5) partially with respect to y^j and using (1.4b), we get

(2.10) $\dot\partial_j (\mathcal{B}_m \mathcal{B}_n H_k) = (\dot\partial_j a_{mn}) H_k + a_{mn} (\dot\partial_j H_k) + (1-n)(\dot\partial_j b_{mn}) y_k + (1-n) b_{mn} g_{jk}$.

Using the commutation formula exhibited by (1.5) for $(\mathcal{B}_n H_k)$ in (2.10) and using (1.9b) and (1.1), we get

(2.11) $\mathcal{B}_m \dot\partial_j (\mathcal{B}_n H_k) = (\dot\partial_j a_{mn}) H_k + a_{mn} H_{jk} + (1-n)(\dot\partial_j b_{mn}) y_k + (1-n) b_{mn} g_{jk}$.

Again applying the commutation formula exhibited by (1.5) for (H_k) in (2.11) and using (1.1), we get

(2.12) $\mathcal{B}_m \mathcal{B}_n H_{jk} = (\dot{\partial}_j a_{mn}) H_k + a_{mn} H_{jk} + (1-n)(\dot{\partial}_j b_{mn}) y_k + (1-n) b_{mn} g_{jk}$.

This shows that

(2.13) $\mathcal{B}_m \mathcal{B}_n H_{jk} = a_{mn} H_{jk} + (1-n) b_{mn} g_{jk}$

if and only if

(2.14) $(\dot{\partial}_j a_{mn}) H_k + (1-n)(\dot{\partial}_j b_{mn}) y_k = 0.$

Thus, we conclude

Theorem2.5. In $G\beta R - BR -$ affinely connected space , $H -$Ricci tensor H_{jk} is non-vanishing if and only if (2.14) holds good.

Transvecting (2.12) by y^k , using (1.3), (1.9a) , (1.9c), (1.4a) , we get

(2.15) $\mathcal{B}_m \mathcal{B}_n H_j = (n-1)(\dot{\partial}_j a_{mn}) H + a_{mn} H_j + (1-n)(\dot{\partial}_j b_{mn}) F^2 + (1-n) b_{mn} y_j$.

Using the condition (2.7) in (2.15), we get

(2.16) $-(\dot{\partial}_j a_{mn}) H + (\dot{\partial}_j b_{mn}) F^2 = 0$

which can be written as

(2.17) $\dot{\partial}_j b_{mn} = \frac{(\dot{\partial}_j a_{mn}) H}{F^2}$.

If the covariant tensor field a_{mn} is independent of y^i, (2.17) shows that the covariant tensor field b_{mn} is independent of y^i . Conversely , if the covariant tensor b_{mn} is independent of y^i , we get $H(\dot{\partial}_j a_{mn}) = 0$. In view theorem 2.2 , the condition $H(\dot{\partial}_j a_{mn}) = 0$ implies $\dot{\partial}_j a_{mn} = 0$,i.e. the covariant tensor field a_{mn} is also independent of y^i . This leads to

Theorem2.6. In $G\beta R - BR -$ affinely connected space , the covariant tensor field b_{mn} is independent of the directional argument**s**.

Suppose the tensor a_{mn} is not independent of y^i and in view of (2.14) and (2.17), we get

(2.18) $\dot{\partial}_j a_{mn} [H_k - \frac{(n-1)}{F^2} H y_k] = 0.$

Transvecting (2.18) by y^m, we get

(2.19) $(\dot{\partial}_j a_{mn}) y^m [H_k - \frac{(n-1)}{F^2} H y_k] = 0,$

which implies

(2.20) $(\dot{\partial}_j a_n - a_{jn}) \left[H_k - \frac{(n-1)}{F^2} H y_k \right] = 0,$

where $a_{mn} y^m = a_n$.

Equation (2.20) has at least one of the following conditions

(2.21) a) $a_{jn} = \dot{\partial}_j a_n$, b) $H_k = \frac{(n-1)}{F^2} H y_k$.

Thus, we conclude

Theorem2.7. In $G\beta R - BR -$ affinely connected space , which the covariant tensor field a_{mn} is not independent of the directional argument at least one of the conditions(2.21a) and (2.21b) hold provided the condition (2.13) holds.

Differentiating the condition (2.3) partially with respect to y^j , using (1.6) and (1.4b), we get

(2.22) $\dot{\partial}_j (\mathcal{B}_m \mathcal{B}_n H^i_{kh}) = (\dot{\partial}_j a_{mn}) H^i_{kh} + a_{mn} H^i_{jkh} + (\dot{\partial}_j b_{mn})(\delta^i_k y_h - \delta^i_h y_k)$
$+ b_{mn} (\delta^i_k g_{jh} - \delta^i_h g_{jk})$.

Using the commutation formula exhibited by (1.5) for $(\mathcal{B}_n H^i_{kh})$ in (2.22) and using (1.1), we get

(2.23) $\mathcal{B}_m (\dot{\partial}_j \mathcal{B}_n H^i_{kh}) = (\dot{\partial}_j a_{mn}) H^i_{kh} + a_{mn} H^i_{jkh} + (\dot{\partial}_j b_{mn})(\delta^i_k y_h - \delta^i_h y_k)$
$+ b_{mn} (\delta^i_k g_{jh} - \delta^i_h g_{jk})$.

Again applying the commutation formula exhibited by (1.5) for (H^i_{kh}) in (2.23) , using (1.6) and (1.1), we get

(2.24) $\mathcal{B}_m \mathcal{B}_n H^i_{jkh} = (\dot{\partial}_j a_{mn}) H^i_{kh} + a_{mn} H^i_{jkh} + (\dot{\partial}_j b_{mn})(\delta^i_k y_h - \delta^i_h y_k) + b_{mn} (\delta^i_k g_{jh} - \delta^i_h g_{jk})$.

This shows that

(2.25) $\mathcal{B}_m \mathcal{B}_n H^i_{jkh} = a_{mn} H^i_{jkh} + b_{mn} (\delta^i_k g_{jh} - \delta^i_h g_{jk})$

if and only if

(2.26) $(\dot{\partial}_j a_{mn}) H^i_{kh} + (\dot{\partial}_j b_{mn})(\delta^i_k y_h - \delta^i_h y_k) = 0$.

Thus, we conclude

Theorem2.8. In $G\beta R - BR -$ affinely connected space , Berwald curvature tensor H^i_{jkh} is generalized $\beta -$birecurrent if and only if (2. 26) holds.

Transvecting (2.26) by y^k , using (1.3) , (1.7a) and (1.4a) , we get

(2.27) $(\dot{\partial}_j a_{mn}) H^i_h - (\dot{\partial}_j b_{mn})(\delta^i_h F^2 - y^i y_h) = 0$.

In view of (2.17) and (2.27), we get

(2.28) $\quad (\dot{\partial}_j a_{mn}) \left[H_h^i - H\left(\delta_h^i - l^i l_h\right) \right] = 0$.

We have at least one of the following conditions

(2.29) $\quad$ a) $\dot{\partial}_j a_{mn} = 0$, $\quad$ b) $H_h^i = H(\delta_h^i - l^i l_h)$.

Putting $H = F^2 R$, $R \neq 0$, equation (2.29b) becomes

(2.30) $\quad H_h^i = F^2 R(\delta_h^i - l^i l_h)$.

Therefore, the space is a Finsler space of scalar curvature.

Thus , we have

Theorem2.9. A $G\beta R-\ BR-$ affinely connected space, for (n >2) is a Finsler space of scalar curvature provided R ≠0 and the covariant tensor filed a_{mn} is not independent of the directional arguments.

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