On a Generalized $\beta R$ – Birecurrent Affinely Connected Space

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Abstract: In the present paper, we introduced a Finsler space whose Cartan’s third curvature tensor $R^i_{jkh}$ satisfies the generalized $\beta R$ – birecurrent property which posses the properties of affinely connected space will be characterized by

$$B_mB_nR^i_{jkh} = a_{mn}R^i_{jkh} + b_{mn}(\delta^i_jg_{jh} - \delta^i_hg_{jh}),$$

where $a_{mn}$ and $b_{mn}$ are non-zero covariant tensors field of second order called birecurrent tensors field, such space is called as a generalized $\beta R$ – birecurrent affine connected space, Ricci tensors $H_{jk}$ and $B_{jk}$ the curvature vector $H_k$ and the curvature scalars $H$ and $R$ of such space are non-vanishing. Some conditions have been pointed out which reduce a generalized $\beta R$ – birecurrent affine connected space $F_n(n > 2)$ into Finsler space of scalar curvature.

Keywords: Finsler space, Generalized $\beta R$ –birecurrent affinely connected space, Finsler space of curvature scalar.

I. Introduction

H.S. Ruse[4] considered a three dimensional Riemannian space having the recurrent of curvature tensor and he called such space as Riemannian space of recurrent curvature. This idea was extended to n-dimensional Riemannian and non- Riemannian space by A.G. Walker [1], Y.C. Wong [9], Y.C. Wong and K. Yano [10] and others. S. Dikshit [8] introduced a Finsler space whose Berwald curvature tensor $H^i_{jkh}$ satisfies the recurrence property in sense of Berwald, F.Y.A. Qasem and A.A.M. Saleem [2] discussed general Finsler space for the $\varphi v$ – curvature tensor $U^i_{jkh}$ satisfies the birecurrence property with respect to Berwald’s coefficient $G^i_{jkh}$ and they called it UBR- Finsler space. P.N. pandey, S. Saxena and A. Goswami [6] introduced a Finsler space whose Berwald curvature tensor $H^i_{jkh}$ satisfies generalized the recurrence property in the sense of Berwald they called such space generalized $H$- recurrent Finsler space . F.Y.A. Qasem and W.H.A. Hadi [3] introduced and studied generalized $\beta R$ – birecurrent Finslerrspace.

An affinely connected space or Berwald space characterized by any one of the two equivalent conditions

1. $G^i_{jkh} = 0$ and $C_{ijk|h} = 0$.

Also it has the following properties

2. $B_k g_{ij} = 0$ and $B_k g^{ij} = 0$.

Berwald covariant derivative of $y^i$ vanishes , i.e.

3. $B_k y^i = 0$.

The vector $y_i$, its associative $y^i$ and the metric tensor $g_{ij}$ given by

4. $y_i y^i = F^2$ and $g_{ij} = \partial y_i = \partial y_i$.

The processes of Berwald’s covariant differentiation and the partial differentiation commute according to

5. $(\partial g_{ij} - B_k g_{ij}) T^{i}_{j} = T^{i}_{j} G^{khr}_{kh} - T^{i}_{k} G^{hij}_{khj}$.

The tensor $H^i_{jkh}$ satisfies the relation

6. $H^i_{jkh} = \partial_j H^i_{kh}$.

The $h(u)$ – torsion tensor $H^i_{kh}$ satisfies

7. $H^i_{kh} y^k = H^i_{kh}$ and $R^i_{jkh} y^j = H^i_{kh}$.

Also we have the following relations

8. $H_{jk} = H^i_{jk}$, $H_k = H^i_{ki}$ and $H = \frac{1}{n-1} H^i_{ij}$.

where $H_{jk}$ and $H$ are called $H$-Ricci tensor [5] and curvature scalar, respectively. Since the contraction of the indices doesn’t effect the homogeneity in $y^i$, hence the tensors $H_{jk}$, $H$, and the scalar $H$ are homogeneous of degree zero, one and two in $y^i$, respectively. The above tensors are also connected by

9. $H_{jk} y^i = H_k$, $H_{jk} = \partial_j H_k$ and $H_k y^k = (n - 1)H$.
The necessary and sufficient condition for a Finsler space $F_n(n > 2)$ to be a Finsler space of curvature scalar is given by

\[
\Delta_i = F^2 R(\delta^i_j - \frac{1}{a} \delta^i_k h_k),
\]

where $R^i_i$ is the deviation of the curvature tensor $H^i_i$.

$R -$ Ricci tensor $R_{jk}$ of the curvature tensor $R^i_{jkh}$, the deviation tensor $R^i_i$ and the curvature scalar $R$ are given by

\[
\begin{align*}
\text{a) } & R^i_{jki} = R^i_{jk}, & \text{b) } & R^i_{jkh} = g^i_{jk} & \text{and} & \text{c) } g^i_{jk} R^i_{jk} = R.
\end{align*}
\]

2. A Generalized $\beta R -$ Birecurrent Affinely Connected Space

A Finsler space whose Cartan's third curvature tensor $R^i_{jkh}$ satisfies the following generalization $\beta R -$ recurrence condition

\[
B_m R^i_{jkh} = \lambda_n R^i_{jkh} + \mu_n (\delta^i_j g_{kh} - \delta^i_k g_{jh}) , \quad R^i_{jkh} \neq 0,
\]

where $\lambda_n$ and $\mu_n$ are non-zero covariant vectors field and called the recurrence vectors field, known as a generalized $\beta R$- recurrence space.

Differentiating (2.1) covariantly with respect to $x^m$ in sense of Berwald and using (1.2a), we get

\[
B_m B_n R^i_{jkh} = a_{mn} R^i_{jkh} + b_{mn} (\delta^i_j g_{kh} - \delta^i_k g_{jh}) ,
\]

where $a_{mn} = B_m A_n + \lambda_n A_m$ and $b_{mn} = \lambda_n \mu_m + B_m \mu_n$ are non-zero covariant tensors field of second order.

Definition 2.1. A Finsler space whose Cartan's third curvature tensor $R^i_{jkh}$ satisfies the condition (2.2) will be called generalized $\beta R$- birecurrent affine connected space, we shall denote it $G \beta R - BR$ - affinely connected space.

Let us consider a $G \beta R - BR$ - affinely connected space. Transvecting the condition (2.2) by $y^i$, using (1.3) and (1.7b), we get

\[
B_m B_n H^i_k = a_{mn} H^i_k + b_{mn} (\delta^i_j y^j - \delta^i_k y^k).
\]

Further, transvecting (2.3) by $y^i$, using (1.3), (1.8a) and (1.4a), we get

\[
B_m B_n H^i_k = a_{mn} H^i_k + b_{mn} (y^j y^j - \delta^i_j F^2).
\]

Thus, we conclude

Theorem 2.1. In $G \beta R - BR$ - affinely connected space, Berwald covariant derivative of second order for the $h(v)$ - torsion tensor $H^i_{kh}$ and the deviation tensor $H^i_k$, given by the conditions (2.3) and (2.4), respectively. Contracting the indices $i$ and $k$ in (2.3) and (2.4), separately and using (1.8b) and (1.8c), we get

\[
\begin{align*}
\text{(2.5)} & \quad B_m B_n H = a_{mn} H + (1 - n) b_{mn} F, \\
\text{(2.6)} & \quad B_m B_n H = a_{mn} H - b_{mn} F^2,
\end{align*}
\]

respectively.

The conditions (2.5) and (2.6), show that the curvature vector $H_k$ and the curvature scalar $H$ cannot vanish, because the vanishing of any one of them would imply $b_{mn} = 0$, a contradiction.

Thus, we conclude

Theorem 2.2. In $G \beta R - BR$ - affinely connected space, the curvature vector $H_k$ and the curvature scalar $H$ are non-vanishing.

Further, transvecting the condition (2.2) by $g^i_k$, using (1.2b) and (1.11b), we get

\[
B_m B_n R^i_k = a_{mn} R^i_k.
\]

Thus, we conclude

Theorem 2.3. In $G \beta R - BR$ - affinely connected space, the deviation tensor $R^i_k$ behaves as birecurrent.

Contracting the indices $i$ and $k$ in the condition (2.2) and using (1.11a), we get

\[
B_m B_n R^i_k = a_{mn} R^i_k + (1 - n) b_{mn} g^i_k.
\]

Transvecting the condition (2.8) by $g^i_k$, using (1.2b) and (1.11c), we get

\[
B_m B_n R = a_{mn} R + (1 - n) b_{mn}.
\]

The conditions (2.8) and (2.9), show that the $R -$ Ricci tensor $R$ and the curvature scalar $R$ cannot vanish, because the vanishing of any one of them would imply $b_{mn} = 0$, a contradiction.

Thus, we conclude

Theorem 2.4. In $G \beta R - BR$ - affinely connected space, $R -$ Ricci tensor $R$ and the curvature scalar $R$ are non-vanishing.

Now, differentiating the condition (2.5) partially with respect to $y^i$ and using (1.4b), we get

\[
\delta^i (B_m B_n H_k) = (\delta a_{mn}) H_k + a_{mn} (\delta H_k) + (1 - n) (\delta b_{mn}) y_k + (1 - n) b_{mn} g^i_k.
\]

Using the commutation formula exhibited by (1.5) for $(B_m H_k)$ in (2.10) and using (1.9b) and (1.1), we get

\[
B_m \delta^i (B_n H_k) = (\delta a_{mn}) H_k + a_{mn} H^i_k + (1 - n) (\delta b_{mn}) y_k + (1 - n) b_{mn} g^i_k.
\]

Again applying the commutation formula exhibited by (1.5) for $(H_k)$ in (2.11) and using (1.1), we get

\[
B_m \delta^i (B_n H_k) = (\delta a_{mn}) H_k + a_{mn} H^i_k + (1 - n) (\delta b_{mn}) y_k + (1 - n) b_{mn} g^i_k.
\]
(2.12) \( B_k B_k H_{jk} = (\delta_j a_{mn}) H_k + a_{mn} H_{jk} + (1 - n)(\delta_j b_{mn}) y_k + (1 - n) b_{mn} g_{jk} \).

This shows that

(2.13) \( B_m B_n H_{jk} = a_{mn} H_{jk} + (1 - n) b_{mn} g_{jk} \)

if and only if

(2.14) \((\delta_j a_{mn}) H_k + (1 - n)(\delta_j b_{mn}) y_k = 0\).

Thus, we conclude

**Theorem 2.5.** In \( G\beta \equiv BR \) — affinely connected space, \( H \) — Ricci tensor \( H_{jk} \) is non-vanishing if and only if (2.14) holds good.

Transvecting (2.12) by \( y^k \), using (1.3), (1.9a), (1.9c), (1.4a), we get

(2.15) \[ B_k B_n H_{jk} = \frac{1}{n-1} \begin{pmatrix} \left( \frac{(n-1)H_k}{F^2} \right) + a_{mn} H_{jk} + (1 - n)(\delta_j b_{mn}) y_k + (1 - n) b_{mn} y_j \end{pmatrix} \]

Using the condition (2.7) in (2.15), we get

(2.16) \[ - (\delta_j a_{mn}) H_k + (\delta_j b_{mn}) F^2 = 0 \]

which can be written as

(2.17) \[ \delta_j b_{mn} = \frac{(\delta_j a_{mn}) H_k}{F^2} \]

If the covariant tensor field \( a_{mn} \) is independent of \( y^i \), (2.17) shows that the covariant tensor field \( b_{mn} \) is independent of \( y^i \). Conversely, if the covariant tensor \( b_{mn} \) is independent of \( y^i \), we get \( H(\delta_j a_{mn}) = 0 \). In view theorem 2.2, the condition \( H(\delta_j a_{mn}) = 0 \) implies \( \delta_j a_{mn} = 0 \), i.e., the covariant tensor field \( a_{mn} \) is also independent of \( y^i \). This leads to

**Theorem 2.6.** In \( G\beta \equiv BR \) — affinely connected space, the covariant tensor field \( b_{mn} \) is independent of the directional arguments.

Suppose the tensor \( a_{mn} \) is not independent of \( y^i \) and in view of (2.14) and (2.17), we get

(2.18) \[ \frac{(\delta_j a_{mn}) H_k}{F^2} H y_k = 0 \].

Transvecting (2.18) by \( y^m \), we get

(2.19) \[ (\delta_j a_{mn}) y^m [H_k - \frac{(n-1) F^2}{F^2} H y_k] = 0 \]

which implies

(2.20) \[ (\delta_j a_n - a_j n) [H_k - \frac{(n-1) F^2}{F^2} H y_k] = 0 \] where \( a_{mn} y^m = a_n \).

Equation (2.20) has at least one of the following conditions

(2.21) \[ \text{a) } a_n = \delta_j a_n, \quad \text{b) } H_k = \frac{(n-1) F^2}{F^2} H y_k \].

Thus, we conclude

**Theorem 2.7.** In \( G\beta \equiv BR \) — affinely connected space, which the covariant tensor field \( a_{mn} \) is not independent of the directional argument at least one of the conditions (2.21a) and (2.21b) provided the condition (2.13) holds.

Differentiating the condition (2.3) partially with respect to \( y^j \), using (1.6) and (1.4b), we get

(2.22) \[ \delta_j B_k B_n H_{jk}^i = \delta_j a_{mn} H_{jk}^i + a_{mn} H_{jk}^i + (\delta_j b_{mn}) (\delta_k^i y_h - \delta_h^i y_k) + b_{mn} (\delta_k^i g_{jh} - \delta_h^i g_{jk}) \]

Using the commutation formula exhibited by (1.5) for \( (B_n H_{jk}^i) \) in (2.22) and using (1.1), we get

(2.23) \[ B_n (\delta_j B_n H_{jk}^i) = \delta_j a_{mn} H_{jk}^i + a_{mn} H_{jk}^i + (\delta_j b_{mn}) (\delta_k^i y_h - \delta_h^i y_k) + b_{mn} (\delta_k^i g_{jh} - \delta_h^i g_{jk}) \]

Again applying the commutation formula exhibited by (1.5) for \( (H_{jk}^i) \) in (2.23), using (1.6) and (1.1), we get

(2.24) \[ B_n B_m H_{jk}^i = (\delta_j a_{mn}) H_{jk}^i + a_{mn} H_{jk}^i + (\delta_j b_{mn}) (\delta_k^i y_h - \delta_h^i y_k) + b_{mn} (\delta_k^i g_{jh} - \delta_h^i g_{jk}) \].

This shows that

(2.25) \[ B_m B_n H_{jk}^i = a_{mn} H_{jk}^i + b_{mn} (\delta_k^i g_{jh} - \delta_h^i g_{jk}) \]

if and only if

(2.26) \[ (\delta_j a_{mn}) H_{jk}^i + (\delta_j b_{mn}) (\delta_k^i y_h - \delta_h^i y_k) = 0 \].

Thus, we conclude

**Theorem 2.8.** In \( G\beta \equiv BR \) — affinely connected space, Berwald curvature tensor \( H_{jk}^i \) is generalized \( \beta \) — bicurrent if and only if (2.26) holds.

Transvecting (2.26) by \( y^k \), using (1.3), (1.7a) and (1.4a), we get

(2.27) \[ (\delta_j a_{mn}) H_{jk}^i - (\delta_j b_{mn}) (\delta_k^i F^2 - y^i y_k) = 0 \].

In view of (2.17) and (2.27), we get
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\begin{equation}
\left( \delta_{ij} a_{mn} \right) \left[ H^i_h - H \left( \delta^i_j - \frac{1}{2} \delta^i_j \right) \right] = 0 .
\end{equation}

We have at least one of the following conditions

\begin{equation}
\begin{align*}
\text{a) } & \delta_{ij} a_{mn} = 0 , & \\
\text{b) } & H^i_h = H \left( \delta^i_j - \frac{1}{2} \delta^i_j \right).
\end{align*}
\end{equation}

Putting $H = F^2 R$, $R \neq 0$, equation (2.29b) becomes

\begin{equation}
H^i_h = F^2 R \left( \delta^i_j - \frac{1}{2} \delta^i_j \right).
\end{equation}

Therefore, the space is a Finsler space of scalar curvature.

Thus, we have

**Theorem 2.9.** A $G \beta R – BR$ – affinely connected space, for $(n \geq 2)$ is a Finsler space of scalar curvature provided $R \neq 0$ and the covariant tensor field $a_{mn}$ is not independent of the directional arguments.

**References**