Qausi Conformal Curvature Tensor on \((LCS)\_n\)-Manifolds

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**ABSTRACT:** In this paper, we focus on quasi-conformal curvature tensor of \((LCS)\_n\)-manifolds. Here we study quasi-conformally flat, Einstein semi-symmetric quasi-conformally flat, \(\xi\)-quasi conformally flat and \(\phi\)-quasi conformally flat \((LCS)\_n\)-manifolds and obtained some interesting results.

**Keywords:** Einstein semi-symmetric, \(\eta\)-Einstein manifold, Lorentzian metric, quasi-conformal curvature tensor, quasi-conformally flat.

I. INTRODUCTION

In 1968, Yano and Sawaki [25] introduced the quasi-conformal curvature tensor given by

\[
\tilde{C}(X,Y)Z = aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY]
- \frac{r}{2n+1} \left( \frac{a}{2n} + 2b \right) [g(Y,Z)X - g(X,Z)Y],
\]

where \(a\) and \(b\) are constants and \(R, S, Q\) and \(r\) are the Riemannian curvature tensor of type \((1,3)\), the Ricci tensor of type \((0,2)\), the Ricci operator defined by \(S(X,Y) = g(QX,Y)\) and scalar curvature of the manifold respectively. If \(a = 1\) and \(b = -\frac{1}{n-2}\), then (1.1) takes the form

\[
\tilde{C}(X,Y)Z = R(X,Y)Z - \frac{1}{n-2} \left[ S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY \right]
- \frac{r}{(n-1)(n-2)} [g(Y,Z)X - g(X,Z)Y] = C(X,Y)Z,
\]

Where \(C\) is the conformal curvature tensor [24]. In [7], De and Matsuyama studied a quasi-conformally flat Riemannian manifold satisfying certain condition on the Ricci tensor. Again Cihan Ozgur and De [5] studied quasi conformal curvature tensor on Kenmotsu manifold and shown that a Kenmotsu manifold is quasi-conformally flat or quasi-conformally semi-symmetric if and only if it is locally isometric to the hyperbolic space. The geometry of quasi-conformal curvature tensor in a Riemannian manifold with different structures were studied by several authors viz. [6, 16, 17, 20].

The present paper is organized as follows: In Section 2 we give the definitions and some preliminary results that will be needed thereafter. In Section 3 we discuss quasi-conformally flat \((LCS)\_n\)-manifolds and it is shown that the manifold is \(\eta\)-Einstein. In Section 4 is devoted to the study of Einstein semi-symmetric quasi-conformally flat \((LCS)\_n\)-manifolds and obtain Qausi Conformal Curvature Tensor on \((LCS)\_n\)-Manifolds 3 that the scalar curvature is constant. In section 5 we consider \(\xi\)-quasi-conformally flat \((LCS)\_n\)-manifolds and proved that the scalar curvature is always constant. Finally, in Section 6, we have shown that a \(\phi\)-quasi conformally flat \((LCS)\_n\)-manifold is an \(\eta\)-Einstein manifold.

II. PRELIMINARIES

The notion of Lorentzian concircular structure manifolds (briefly \((LCS)\_n\)-manifolds) was introduced by A.A. Shaikh [18] in 2003. An \(n\)-dimensional Lorentzian manifold \(M\) is a smooth connected paracompact Hausdorff manifold with a Lorentzian metric \(g\), that is, \(M\) admits a smooth symmetric tensor field \(g\) of type \((0,2)\) such that for each point \(p \in M\), the tensor \(g_p: T_pM \times T_pM \rightarrow \mathbb{R}\) is a non-degenerate inner product of signature \((-+,++,...,+))\), where \(T_pM\) denotes the tangent vector space of \(M\) at \(p\) and \(R\) is the real number space.

**Definition 2.1** In a Lorentzian manifold \((M, g)\), a vector field \(P\) defined by \(g(X, P) = A(X)\), for any vector field \(X \in \chi(M)\) is said to be a concircular vector field if

\[
(\nabla_X A)(Y) = a[g(X,Y) + \omega(X)A(Y)],
\]

Where \(a\) is a non-zero scalar function, \(A\) is a 1-form and \(\omega\) is a closed 1-form.

Let \(M\) be a \(n\)-dimensional Lorentzian manifold admitting a unit timelike concircular vector field \(\xi\), called the characteristic vector field of the manifold. Then we have

\[
g(\xi, \xi) = -1.
\]

Since \(\xi\) is a unit concircular vector field, there exists a non-zero 1-form \(\eta\) such that

\[
g(X, \xi) = \eta(X),
\]

the equation of the following form holds
\[(\nabla_X \eta)(Y) = \alpha [g(X,Y) + \eta(X)\eta(Y)], (\alpha \neq 0) \quad (2.3)\]

for all vector fields \(X, Y\), where \(r\) denotes the operator of covariant differentiation with respect to Lorentzian metric \(g\) and \(\alpha\) is a non-zero scalar function satisfying
\[
\nabla_X \alpha = (X\alpha) = da(X) = \rho \eta(X), \quad (2.4)
\]

\(\rho\) being a certain scalar function given by \(\rho = -\langle \xi, \alpha \rangle\). If we put
\[
\phi X = -\frac{1}{\alpha} \nabla_X \xi, \quad (2.5)
\]

Then from (2.3) and (2.4), we have
\[
\phi X = X + \eta(X)\xi, \quad (2.6)
\]

from which it follows that \(\phi\) is a symmetric \((1,1)\) tensor. Thus the Lorentzian manifold \(M\) together with the unit time like concircular vector field \(\xi\), its associated 1-form \(\eta\) and \((1,1)\) tensor field \(\phi\) is said to be a Lorentzian concircular structure manifold (briefly \((LCS)_n\)-manifold). Especially, if we take \(\alpha = 1\), then we can obtain the Lorentzian para-Sasakian structure of Matsumoto [12]. In a \((LCS)_n\)-manifold, the following relations hold ([18], [19]):

\[
\eta(\xi) = -1, \quad \phi \xi = 0, \quad \eta(\phi X) = 0, \quad (2.7)
\]

\[
g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \quad (2.8)
\]

\[
\eta(R(X,Y)Z) = (a^2 - \rho)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \quad (2.9)
\]

\[
R(X,Y)\xi = (a^2 - \rho)[\eta(Y)X - \eta(X)Y]. \quad (2.10)
\]

\[
(\nabla_X \phi)(Y) = a[g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X], \quad (2.11)
\]

\[
S(X,\xi) = (n-1)(a^2 - \rho)\eta(X), \quad (2.12)
\]

\[
S(\phi X, \phi Y) = S(X,Y) + (n-1)(a^2 - \rho)\eta(X)\eta(Y), \quad (2.13)
\]

\[
Q^{\xi} = (n-1)(a^2 - \rho)\xi, \quad (2.14)
\]

for any vector fields \(X,Y,Z\), where \(R,S\) denote respectively the curvature tensor and the Ricci tensor of the manifold.

**Definition 2.2** An \((LCS)_n\)-manifold \(M\) is said to be Einstein if its Ricci tensor \(S\) is of the form
\[
S(Y) = ag(X,Y), \quad (2.15)
\]

for any vector fields \(X\) and \(Y\), where \(a\) is a scalar function.

### III. QUASI-CONFORMALLY FLAT \((LCS)_n\)-MANIFOLDS

Let us consider quasi-conformally flat \((LCS)_n\)-manifolds, i.e., \(\tilde{C}(X,Y)Z = 0\). Then from (1.1), we have
\[
R(X,Y)Z = \frac{b}{a^n} [S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY]
\]

\[
+ \frac{a}{an} (a + 2b) [g(Y,Z)X - g(X,Z)Y]. \quad (3.1)
\]

Taking inner product of (3.1) with respect to \(W\), we get
\[
R(X,Y,Z,W) = \frac{b}{a^n} [S(Y,Z)g(X,W) - S(X,Z)g(Y,W) + g(Y,Z)S(X,W) - g(X,Z)S(Y,W)]
\]

\[
+ \frac{a}{an} (a + 2b) [g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]. \quad (3.2)
\]

Also from (2.9), we have
\[
R(\xi, Y, Z, \xi) = -(a^2 - \rho)[g(Y,Z) + \eta(Y)\eta(Z)]. \quad (3.3)
\]

Putting \(X = W = \xi\) in (3.2) becomes
\[
R(\xi, Y, Z, \xi) = \frac{b}{a^n} [S(Y,Z)g(\xi, \xi) - S(\xi,Z)g(Y,\xi) + g(Y,Z)S(\xi,\xi) - g(\xi,Z)S(Y,\xi)]
\]

\[
+ \frac{a}{an} (a + 2b) [g(Y,Z)g(\xi,\xi) - g(\xi,Z)g(Y,\xi)]. \quad (3.4)
\]

By virtue of (2.1), (2.2), (2.12) and (3.3) equation (3.4) yields
\[
S(Y,Z) = Mg(Y,Z) + N\eta(Y)\eta(Z), \quad (3.5)
\]

Where
\[
M = \frac{r}{bn} \left(\frac{a}{n-1} + 2b\right) - \frac{a^2 - \rho}{b} \left[\frac{a}{n-1} + b(n-1)\right],
\]

\[
N = \frac{r}{bn} \left(\frac{a}{n-1} + 2b\right) - \frac{a^2 - \rho}{b} \left[\frac{a}{n-1} + b(n-1)\right].
\]

Thus we state:

**Theorem 3.1** A quasi-conformally flat \((LCS)_n\)-manifold is an \(\eta\)-Einstein manifold.
IV. EINSTEIN SEMI-SYMMETRIC QUASI-CONFORMALLY FLAT \((LCS)_n\)-MANIFOLDS

The Einstein tensor is given by

\[
E(X,Y) = S(X,Y) - \frac{r}{2}g(X,Y),
\]

(4.1)

Where \(S\) is the Ricci tensor and \(r\) is the scalar curvature. An \(n\)-dimensional quasi-conformally flat \((LCS)_n\)-manifold is said to be Einstein semi-symmetric if,

\[
R(X,Y) \cdot E(Z,W) = 0.
\]

(4.2)

By using equation (3.5), we have

\[
QX = MX + \eta(X)\xi.
\]

(4.3)

Substituting (3.5) and (4.3) in (3.1), we get

\[
R(X,Y)Z = A[g(Y,Z)X - g(X,Z)Y] + B[\eta(Y)\eta(Z)X - \eta(X)\eta(Y)Y + g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] = 0,
\]

(4.4)

Where

\[
A = \frac{a^2 - \rho}{b}[a + b(n-1) - \frac{a}{mn}(\frac{a}{n-1} + 2b)]
\]

\[
B = \frac{r}{bm}(\frac{a}{n-1} + 2b) - \frac{a^2 - \rho}{b}[a + 2b(n-1)].
\]

Now, we consider the quasi-conformally flat \((LCS)_n\)-manifold which is Einstein semi-symmetric i.e.,

\[
R(X,Y) \cdot E(Z,W) = 0.
\]

Then we have

\[
E(R(X,Y)Z, U) + E(Z, R(X,Y)U) = 0.
\]

(4.5)

By virtue of (4.1), equation (4.5) becomes

\[
S(R(X,Y)Z, U) - \frac{r}{2}g(R(X,Y)Z, U) + S(Z, R(X,Y)U) - \frac{r}{2}g(Z, R(X,Y)U) = 0.
\]

(4.6)

Using (3.5) in (4.6), we get

\[
S(R(X,Y)Z, U) - \frac{r}{2}g(R(X,Y)Z, U) + S(Z, R(X,Y)U) - \frac{r}{2}g(Z, R(X,Y)U) = 0.
\]

(4.7)

Put \(Z = \xi\) in (4.7), we obtain

\[
(M - \frac{r}{2})g(R(X,Y)\xi, U) + (M - \frac{r}{2})\eta(R(X,Y)U) + N\eta(R(X,Y)\xi)\eta(U) - N\eta(R(X,Y)U) = 0.
\]

(4.8)

By virtue of (4.4), above equation becomes

\[
N(B - A)[g(Y, U)\eta(X) - g(X, U)\eta(Y)] = 0.
\]

(4.9)

Putting \(Y = \xi\) in (4.9), we get

\[
N[g(X, U) + \eta(U)\eta(X)] = 0.
\]

(4.10)

Again putting \(U = QW\) in (4.10), we have

\[
N[S(X, W) + \eta(QW)\eta(X)] = 0.
\]

(4.11)

Using (4.3) in (4.11), gives

\[
N[S(X, W) + (M - N)\eta(W)\eta(X)] = 0.
\]

(4.12)

Either \(N = 0\) or \(S(X, W) + (M - N)\eta(W)\eta(X) = 0\).

As \(N \neq 0\), we have

\[
S(X, W) + (M - N)\eta(W)\eta(X) = 0.
\]

(4.13)

Put \(X = W = e_i\) in (4.13) and taking summation over \(i\), \(1 \leq i \leq n\), we get

\[
r = (n-1)(\alpha^2 - \rho).
\]

(4.14)

Hence we can state the following:

**Theorem 4.2** In an Einstein semi-symmetric quasi-conformally flat \((LCS)_n\)-manifold, the scalar curvature is constant.

V. \(\xi\)-QUASI CONFORMALLY FLAT \((LCS)_n\)-MANIFOLDS

A Let \(M\) be an \(n\)-dimensional \(\xi\)-quasi conformally flat \((LCS)_n\)-manifold. i.e.,

\[
\tilde{C}(X,Y)\xi = 0, \quad \forall X, Y \in TM.
\]

(5.1)

Putting \(Z = \xi\) in (3.1), we get

\[
\tilde{C}(X,Y)\xi = aR(X,Y)\xi + b[S(X,\xi)X - S(X,\xi)Y + g(Y,\xi)QX - g(X,\xi)QY]
\]

\[
- \frac{r}{n}\left[\frac{a}{n-1} + 2b\right][g(Y,\xi)X - g(X,\xi)Y].
\]

(5.2)

Since \(\tilde{C}(X,Y)\xi = 0\), we have

\[
aR(X,Y)\xi = \frac{r}{n}\left[\frac{a}{n-1} + 2b\right][g(Y,\xi)X - g(X,\xi)Y]
\]

\[
- b[S(X,\xi)X - S(X,\xi)Y + g(Y,\xi)QX - g(X,\xi)QY].
\]

(5.3)

Using (2.2), (2.10) and (2.12) in (5.3) becomes

\[
a(\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y] = \frac{r}{n}\left[\frac{a}{n-1} + 2b\right][\eta(Y)X - \eta(X)Y]
\]

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Again putting \( Y = \xi \) in (5.4), we get
\[-a(a^2 - \rho)[X + \eta(X)\xi] = -\frac{r}{n}\left(\frac{a}{n-1} + 2b\right)[X + \eta(X)\xi] + 2(n-1)(a^2 - \rho)b[X + \eta(X)\xi].\] (5.5)

Taking inner product of above equation with respect to \( W \), we obtain
\[-a(a^2 - \rho)[g(X, W) + \eta(X)\eta(W)] = -\frac{r}{n}\left(\frac{a}{n-1} + 2b\right)[g(X, W) + \eta(X)\eta(W)] + 2(n-1)(a^2 - \rho)b[g(X, W) + \eta(X)\eta(W)].\]

Put \( X = W = e_i \) in (5.6) and taking summation over \( i, 1 \leq i \leq n \), we get
\[r = \frac{a}{n-1} + 2b\]
\[-a + 2b(n-1)(a^2 - \rho).\]

Hence we can state:

**Theorem 5.3** In an \( \xi \)-quasi conformally flat \((LCS)_n\)-manifold, the scalar curvature is constant.\( \cdots \)

**VI. \( \phi \)-QUASI CONFORMALLY FLAT \((LCS)_n\)-MANIFOLDS**

A Let \( M \) be an \( n \)-dimensional \((LCS)_n\)-manifold is said to be \( \phi \)-quasi conformally flat if it satisfies
\[\phi^2\mathcal{C}(\phi X, \phi Y)\phi Z = 0.\]
(6.1)

**Theorem 6.4** An \( n \)-dimensional \( \phi \)-quasi conformally flat \((LCS)_n\)-manifold is an \( \eta \)-Einstein manifold.

**Proof:** Let us consider \( \phi \)-quasi conformally flat \((LCS)_n\)-manifold i.e., \( \phi^2\mathcal{C}(\phi X, \phi Y)\phi Z = 0 \). It can be easily see that
\[g(\mathcal{C}(\phi X, \phi Y)\phi Z, \phi W) = 0.\]
(6.2)

By virtue of (1.1), we have
\[ag(R(\phi X, \phi Y)\phi Z, \phi W) = -b[S(\phi Y, \phi Z)g(\phi X, \phi W)] - S(\phi X, \phi Z)g(\phi Y, \phi W) + S(\phi X, \phi W)g(\phi Y, \phi Z) - S(\phi Y, \phi W)g(\phi X, \phi Z)\]
\[+ \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)[g(\phi Y, \phi Z)g(\phi X, \phi W) - g(\phi X, \phi Z)g(\phi Y, \phi W)].\]
(6.3)

Let \( \{e_1, e_2, \ldots, e_{n-1}, \xi\} \) be a local orthonormal basis of vector fields in \( M \).

As \( \{\phi(e_1), \phi(e_2), \ldots, \phi(e_{n-1}), \xi\} \) is also a local orthonormal basis, if we put \( X = W = e_i \) in (6.3) and sum up with respect to \( i \), then we have
\[a \sum_{i=1}^{n-1} g(R(\phi(e_i), \phi Y)\phi Z, \phi(e_i))\]
\[= \sum_{i=1}^{n-1} \left[S(\phi Y, \phi Z)g(\phi(e_i), \phi(e_i)) - S(\phi(e_i), \phi Z)g(\phi Y, \phi(e_i))\right]
\[+ S(\phi(e_i), \phi(e_i))g(\phi Y, \phi Z) - S(\phi Y, \phi(e_i))g(\phi(e_i), \phi Z)\]
\[+ \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)\sum_{i=1}^{n-1} [g(\phi Y, \phi Z)g(\phi(e_i), \phi(e_i)) - g(\phi(e_i), \phi Z)g(\phi Y, \phi(e_i))].\]
(6.4)

It can be easily verify that
\[\sum_{i=1}^{n-1} g(R(\phi(e_i), \phi Y)\phi Z, \phi(e_i)) = S(\phi Y, \phi Z) + g(\phi Y, \phi Z),\]
(6.5)
\[\sum_{i=1}^{n-1} S(\phi Y, \phi(e_i))g(\phi(e_i), \phi Z) = S(\phi Y, \phi Z),\]
(6.6)
\[\sum_{i=1}^{n-1} S(\phi(e_i), \phi(e_i)) = r + (n-1)(\alpha^2 - \rho),\]
(6.7)
\[\sum_{i=1}^{n-1} g(\phi(e_i), \phi(e_i)) = n - 1,\]
(6.8)
\[\sum_{i=1}^{n-1} g(\phi(e_i), \phi Z)g(\phi Y, \phi(e_i)) = g(\phi Y, \phi Z).\]
(6.9)

Using (6.5) to (6.9) in (6.4), we get
\[S(Y, Z) = Mg(Y, Z) + Nn(Y)\eta(Z),\]

Where

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\[ M = \frac{r(n-2)}{n} \left( \frac{a}{n-1} + 2b \right) + (n-1)(a^2 - \rho) + r - a, \]
\[ N = \frac{r(n-2)}{n} \left( \frac{a}{n-1} + 2b \right) + r + (a^2 - \rho)[(n-1)(1-b) - an]. \]

Hence the proof

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