Certain Generalized Birecurrent Tensors In $K^h - \text{GBR-} F_n$.

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Abstract: We presented a Finsler space $F_n$ whose Cartan’s fourth curvature tensor $K_{jk h}^i$ satisfies

$$K_{jk h}^i|_{m} = \lambda_{r} K_{jk h}^i|_{m} + b_{r m} K_{jk h}^i , \quad K_{jk h}^i \neq 0 ,$$

where $\lambda_{r}$ and $b_{r m}$ are non-zero covariant vector field and covariant tensor field of second order, respectively. Such space is called as $K^h$-generalized birecurrent space and denoted briefly by $K^h - \text{GBR-} F_n$. In the present paper we shall obtain some generalized birecurrent tensor in an $K^h - \text{GBR-} F_n$.

Keywords: Finsler space, $K^h$ - Generalized birecurrent Finsler space, Ricci tensor.

I. Introduction

Let $F_n$ be an $n$-dimensional Finsler space equipped with the metric function $F(x, y)$ satisfying the request conditions [7].

Cartan’s second kind covariant differentiation form arbitrary vector field $x^i$ with respect to $x^h$ is given by [3],[4]

$$X_{jk}^i := \partial_{j} x^i - \left( \partial_{r} x^j \right) G_{rk}^i + X^i \Gamma^r_{jk}.$$

M. Motsumoto [5],[6] calls this derivative as $h-$c covariant derivative.

The vector $y^i$ and the metric tensor $g_{ij}$ and its associate satisfies the following relations

$$\begin{align*}
\text{(1.1)} & \quad a) \quad y^i_{jk} = 0 \quad , \quad b) \quad g_{ij|k} = 0 \quad \text{and} \quad c) \quad g_{i|jk} = 0. \\
\text{Also satisfies the following relation} & \\
\text{(1.3)} & \quad C_{ijk} g^{jk} = C_i. \\
\text{The} (v) h^-\text{-torsion tensor} C_{ijk} \text{is the associate tensor of the} (h) h^-\text{-vector field} C_{ijk} \text{and defined by} & \\
\text{(1.4)} & \quad C_{ik} := g_{ij} C_{ijk}. \\
\text{The tensor} C_{ik} \text{is positively homogeneous of degree} -1 \text{in} y^i \text{and symmetric in all its indices}. \text{By using Euler’s theorem on homogeneous properties, this tensor satisfies the following} & \\
\text{(1.5)} & \quad P_{ijk}^j = \left( \partial_{j} \Gamma_{ik}^{r*} \right) y^i = \Gamma_{ijk}^{r*} y^h. \\
\text{Berwald curvature tensor} H_{jk h}^{i} \text{and the} (h) \text{- torsion tensor} H_{jk h}^{i} \text{are related by} & \\
\text{(1.6)} & \quad H_{jk h}^{i} y^j = H_{jk}^{i}. \\
\text{The deviation tensor} H_{jk}^{i} \text{is positively homogeneous of degree two in} y^i \text{and satisfies} & \\
\text{(1.7)} & \quad H_{jk h}^{i} y^j = H_{jk}^{i}. \\
\text{Cartan’s fourth curvature tensor} K_{jk h}^{i} \text{satisfies the following identity known as \textit{Bianchi identity}} & \\
\text{(1.8)} & \quad K_{ijk h}^{i} + K_{ij k h}^{i} + K_{i j h k}^{i} + y^r \left( \left( \partial_{s} \Gamma_{ijk}^{r*} \right) K_{r}^{s} + \left( \partial_{s} \Gamma_{r}^{i} \right) K_{r h k}^{s} + \left( \partial_{s} \Gamma_{r j k}^{*} \right) K_{r}^{s} \right) = 0. \\
\text{The associate tensor} K_{ijk h}^{i} \text{of the curvature tensor} K_{jk h}^{i} \text{is given by} & \\
\text{(1.9)} & \quad K_{ijk h}^{i} := g_{jk} K_{jk h}^{i}. \\
\text{The tensor} K_{ijk h}^{i} \text{also satisfies the condition} & \\
\text{(1.10)} & \quad K_{ijk h}^{i} y^j = H_{jk h}^{i}. 
\end{align*} $$

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(1.11) \( K^i_{jk} = K_{jk} \)

and

(1.12) \( H^i_{jkh} - K^i_{jkh} = P^i_{jkh} + P^i_{jkh}P^h_{rk} - h/k^* \).

* \(- h/k^* \) means the subtraction from the former term by interchange the indices \( h \) and \( k \).

We know the identity [7]
\[(2.16) \quad K_j = H_j - H_j^I C_i.\]
Differentiating (2.16) covariantly with respect to \(x^\ell\) in the sense of Cartan, we get
\[(2.17) \quad K_{j|\ell} = H_{j|\ell} - (H_j^I C_i)_{|\ell}.\]
Differentiating (2.17) covariantly with respect to \(x^m\) in the sense of Cartan, we get
\[(2.18) \quad K_{j|m} = H_{j|m} - (H_j^I C_i)_{|m}.\]
Using (2.4) and (2.14) in (2.18), we obtain
\[(2.19) \quad K_{j|m} = \lambda \{H_{j|m} - (H_j^I C_i)_{|m}\} + b_{|m} \{H_j - (H_j^I C_i)\}.\]
Putting (2.16) and (2.17) in (2.19), we get
\[(2.20) \quad K_{j|m} = \lambda K_j + b_{|m} K_j.\]
Thus, we conclude

**Theorem 2.2.** In \(K^h - GBR - F_n\), the vector \(K_j\) is generalized birecurrent.

Also, we have the identity [7]
\[(2.21) \quad R_j = K_j + C_{j|r} H_r^I.\]
Differentiating (2.21) covariantly with respect to \(x^\ell\) in the sense of Cartan, we get
\[(2.22) \quad R_{j|\ell} = K_{j|\ell} + (C_{j|\ell} H_r^I)_{|\ell}.\]
Differentiating (2.22) covariantly with respect to \(x^m\) in the sense of Cartan, we get
\[(2.23) \quad R_{j|m} = K_{j|m} + (C_{j|m} H_r^I)_{|m}.\]
Using (2.15) and (2.20) in (2.23), we get
\[(2.24) \quad R_{j|m} = \lambda \{K_{j|m} + (C_{j|m} H_r^I)_{|m}\} + b_{|m} \{K_j + (C_{j|\ell} H_r^I)\}.\]
Putting (2.21) and (2.22) in (2.24), we get
\[(2.25) \quad R_{j|m} = \lambda R_j + b_{|m} R_j.\]
Thus, we conclude

**Theorem 2.3.** In \(K^h - GBR - F_n\), the vector \(R_j\) is generalized birecurrent.

We have Cartan’s fourth curvature tensor \(K^{ij}_{k|\ell}\), \(v(hv)\) torsion tensor \(P^i_{jk}\) and Berwald curvature tensor \(H^i_{jkh}\) are connected by the formula (1.12).

Differentiating (1.12) covariantly with respect to \(x^\ell\) in the sense of Cartan, we get
\[(2.25) \quad H^{ij}_{k|\ell} = K^{ij}_{k|\ell} = \{P^i_{jk|h} + P^r_{jk} P^i_{r|h} - h/k\}_{|\ell}.\]
Differentiating (2.25) covariantly with respect to \(x^m\) in the sense of Cartan, we get
\[(2.26) \quad H^{ij}_{k|m} = K^{ij}_{k|m} = \{P^i_{jk|h} + P^r_{jk} P^i_{r|h} - h/k\}_{|m}.\]
Using (2.1) and if Berwald curvature tensor \(H^i_{jkh}\) is generalized birecurrent, (2.26) reduces to
\[(2.27) \quad \lambda \{H^{ij}_{k|m} - K^{ij}_{k|m}\} + b_{|m} \{H^i_{jkh} - K^{ij}_{k|h}\} = \{P^i_{jk|h} + P^r_{jk} P^i_{r|h} - h/k\}_{|m}.\]
Putting (1.12) and (2.25) in (2.27), we get
\[(2.28) \quad \{P^i_{jk|h} + P^r_{jk} P^i_{r|h} - h/k\}_{|m} = \lambda \{h^i_{jk|h} + P^r_{jk} P^i_{r|h} - h/k\}_{|m} + b_{|m} \{P^i_{jk|h} + P^r_{jk} P^i_{r|h} - h/k\}.\]
Thus, we conclude

**Theorem 2.4.** In \(K^h - GBR - F_n\), the tensor \(\{P^i_{jk|h} + P^r_{jk} P^i_{r|h} - h/k\}\) is generalized birecurrent [provided Berwald curvature tensor \(H^i_{jkh}\) is generalized birecurrent].

We know the curvature tensor \(K^{ij}_{k|h}\) satisfies [5] the identity
\[(2.28) \quad K^{ij}_{k|h} - K^{ij}_{k|hi} = H^i_{ij} C_{rk} - H^r_{kj} C_{rij} + H^r_{ki} C_{rjh} - H^r_{ij} C_{rkh} - H^r_{kj} C_{rjh} + H^r_{ki} C_{rjh}.\]
Differentiating (2.28) covariantly with respect to \(x^\ell\) in the sense of Cartan, we get
\[(2.29) \quad K^{ij}_{k|\ell} - K^{ij}_{k|\ell hi} = \{H^i_{ij} C_{rk} - H^r_{kj} C_{rij} + H^r_{ki} C_{rjh} - H^r_{ij} C_{rkh} - H^r_{kj} C_{rjh} + H^r_{ki} C_{rjh}\}_{|\ell}.\]
Differentiating (2.29) covariantly with respect to \(x^m\) in the sense of Cartan, we get
\[(2.30) \quad K^{ij}_{k|m} - K^{ij}_{k|m hi} = \{H^i_{ij} C_{rk} - H^r_{kj} C_{rij} + H^r_{ki} C_{rjh} - H^r_{ij} C_{rkh}\}_{|m}.\]
Using (2.2) in (2.30), we get
\[
\lambda(t) (K_{ijkl \ell} - K_{jkhi \ell}) + b_{\ell m} (K_{ijkl} - K_{jkhi}) = (H^i_{jk} C_{rk} - H^i_{hk} C_{rij}) + H^i_{ik} C_{rj h} - H^i_{ij} C_{rkh} - H^i_{jk} C_{rhi} + H^i_{ki} C_{rj h})_{\ell m}.
\]
Putting (2.28) and (2.29) in (2.31), we get
\[
\lambda(t) (H^i_{jk} C_{rk} - H^i_{hk} C_{rij} + H^i_{ik} C_{rj h} - H^i_{ij} C_{rkh} - H^i_{jk} C_{rhi} + H^i_{ki} C_{rj h})_{\ell m} + b_{\ell m} (H^i_{jk} C_{rk} - H^i_{hk} C_{rij} + H^i_{ik} C_{rj h} - H^i_{ij} C_{rkh} - H^i_{jk} C_{rhi} + H^i_{ki} C_{rj h})_{\ell m}.
\]
Transvecting (2.32) by \(y^j\), using (1.1a), (1.2) and (1.6), we get
\[
\lambda(t) (H^i_{jk} C_{rk} - H^i_{hk} C_{rij} + H^i_{ik} C_{rj h} - H^i_{ij} C_{rkh} - H^i_{jk} C_{rhi} + H^i_{ki} C_{rj h})_{\ell m} + b_{\ell m} (H^i_{jk} C_{rk} - H^i_{hk} C_{rij} + H^i_{ik} C_{rj h} - H^i_{ij} C_{rkh} - H^i_{jk} C_{rhi} + H^i_{ki} C_{rj h}).
\]
Transvecting (2.33) by \(g^{pr}\), using (1.1c) and (1.4), we get
\[
\lambda(t) (H^i_{jk} C_{rk} - H^i_{hk} C_{rij} + H^i_{ik} C_{rj h} - H^i_{ij} C_{rkh} - H^i_{jk} C_{rhi} + H^i_{ki} C_{rj h})_{\ell m} + b_{\ell m} (H^i_{jk} C_{rk} - H^i_{hk} C_{rij} + H^i_{ik} C_{rj h} - H^i_{ij} C_{rkh} - H^i_{jk} C_{rhi} + H^i_{ki} C_{rj h}).
\]
Thus, we conclude

**Theorem 2.5.** In \(K^h \# GBR \sim F_a\), the tensors \((H^i_{jk} C_{rk} - H^i_{hk} C_{rij} + H^i_{ik} C_{rj h} - H^i_{ij} C_{rkh} - H^i_{jk} C_{rhi} + H^i_{ki} C_{rj h})_{\ell m}\) are all generalized birecurrent.

Using (1.10) in (2.34), we get
\[
K_{ijkl} + K_{iklj} + K_{ikjl} = -2 y^i \left[ C_{ij s} K^s_{rkh} + C_{iks} K^s_{rjh} + C_{iks} K^s_{rjh} \right].
\]
Differentiating (2.35) covariantly with respect to \(x^i\) in the sense of Cartan, we get
\[
K_{ijkl \ell} + K_{iklj \ell} + K_{ikjl \ell} = -2 \left[ C_{ij s} K^s_{rkh} + C_{iks} K^s_{rjh} + C_{iks} K^s_{rjh} \right]_{\ell}. \]
Differentiating (2.36) covariantly with respect to \(x^m\) in the sense of Cartan, we get
\[
K_{ijkl \ell m} + K_{iklj \ell m} + K_{ikjl \ell m} = -2 \left[ C_{ij s} K^s_{rkh} + C_{iks} K^s_{rjh} + C_{iks} K^s_{rjh} \right]_{\ell m}. \]
Using (2.2), (2.35) and (2.36) in (2.37), we get
\[
\lambda(t) (C_{ij s} H^s_{rkh} + C_{iks} H^s_{rjh} + C_{iks} H^s_{rjh})_{\ell m} = \lambda(t) (C_{ij s} H^s_{rkh} + C_{iks} H^s_{rjh} + C_{iks} H^s_{rjh})_{\ell m} + b_{\ell m} (C_{ij s} H^s_{rkh} + C_{iks} H^s_{rjh} + C_{iks} H^s_{rjh})_{\ell m}.
\]
Transvecting (2.38) by \(y^i\), using (1.1a), (1.2) and (1.6), we get
\[
\lambda(t) (C_{ij s} H^s_{rkh} + C_{iks} H^s_{rjh} + C_{iks} H^s_{rjh})_{\ell m} + b_{\ell m} (C_{ij s} H^s_{rkh} + C_{iks} H^s_{rjh} + C_{iks} H^s_{rjh})_{\ell m}.
\]
Transvecting (2.39) by \(g^{pr}\), using (1.1c) and (1.4), we get
\[
\lambda(t) (C_{ij s} H^s_{rkh} + C_{iks} H^s_{rjh} + C_{iks} H^s_{rjh})_{\ell m} + b_{\ell m} (C_{ij s} H^s_{rkh} + C_{iks} H^s_{rjh} + C_{iks} H^s_{rjh}).
\]
Thus, we conclude

**Theorem 2.6.** In \(K^h \# GBR \sim F_a\), the tensors \((C_{ij s} H^s_{rkh} + C_{iks} H^s_{rjh} + C_{iks} H^s_{rjh})_{\ell m}\) and \((C_{ij s} H^s_{rkh} + C_{iks} H^s_{rjh} + C_{iks} H^s_{rjh})_{\ell m}\) are all generalized birecurrent.

Differentiating (1.7) covariantly with respect to \(x^m\) in the sense of Cartan, we get
\[
K_{j^k i^l k^h} \ell m + K_{j^k i^l h^k} \ell m + y^{r} \left[ (\partial_{i^j r} K^s_{k h} + (\partial_{i^j r} K^s_{k h})_{\ell m} + (\partial_{i^j r} K^s_{k h})_{\ell m} + (\partial_{i^j r} K^s_{k h})_{\ell m} + (\partial_{i^j r} K^s_{k h})_{\ell m} + (\partial_{i^j r} K^s_{k h})_{\ell m} = 0.
\]
Using (2.1) in (2.40), we get
\[
\lambda(t) K_{j^k i^l k^h} \ell m + \lambda(k K^s_{i^j r} + (\partial_{i^j r} K^s_{k h})_{\ell m} + (\partial_{i^j r} K^s_{k h})_{\ell m} + (\partial_{i^j r} K^s_{k h})_{\ell m} + (\partial_{i^j r} K^s_{k h})_{\ell m} + (\partial_{i^j r} K^s_{k h})_{\ell m} + (\partial_{i^j r} K^s_{k h})_{\ell m} = 0.
\]
If Cartan’s fourth curvature tensor $K^i_{jkh}$ is recurrent, (2.41) becomes

\begin{equation}
\begin{aligned}
\lambda_j \lambda_k K^i_{jkh} + \lambda_k \lambda_m K^i_{jkh} + \lambda_k \lambda_m K^i_{jkh} + b_{\ell m} K^i_{jkh} + b_{bm} K^i_{jkh} + b_{\ell m} K^i_{jkh} \\
+ \lambda_m y^r (\partial_j \Gamma^r_{i\ell}) K^s_{rhe} + (\partial_j \Gamma^r_{i\ell}) K^s_{rhe} + (\partial_j \Gamma^r_{i\ell}) K^s_{rhe} \\
+ y^r \left\{ (\partial_j \Gamma^r_{i\ell}) |m K^s_{rhe} + (\partial_j \Gamma^r_{i\ell}) |m K^s_{rhe} + (\partial_j \Gamma^r_{i\ell}) |m K^s_{rhe} \right\} = 0.
\end{aligned}
\end{equation}

Putting (1.7) in (2.42), we get

\begin{equation}
\lambda_j \lambda_m K^i_{jkh} + \lambda_k \lambda_m K^j_{jkh} + \lambda_k \lambda_m K^i_{jkh} + b_{\ell m} K^i_{jkh} + b_{bm} K^i_{jkh} + b_{bm} K^i_{jkh} \\
- \lambda_m (K^i_{jkh}) + K^i_{jkh} + K^i_{jkh} + y^r \left\{ (\partial_j \Gamma^r_{i\ell}) |m K^s_{rhe} + (\partial_j \Gamma^r_{i\ell}) |m K^s_{rhe} + (\partial_j \Gamma^r_{i\ell}) |m K^s_{rhe} \right\} = 0.
\end{equation}

which can be written as

\begin{equation}
\begin{aligned}
b_{\ell m} K^i_{jkh} + b_{bm} K^i_{jkh} + b_{bm} K^i_{jkh} + + y^r \left\{ (\partial_j \Gamma^r_{i\ell}) |m K^s_{rhe} + (\partial_j \Gamma^r_{i\ell}) |m K^s_{rhe} + (\partial_j \Gamma^r_{i\ell}) |m K^s_{rhe} \right\} = 0
\end{aligned}
\end{equation}

Transvecting (2.43) by $y^i$, using (1.1a), (1.10) and (1.5), we get

\begin{equation}
\begin{aligned}
b_{\ell m} H^i_{kh} + b_{bm} H^i_{kh} + b_{bm} H^i_{kh} + + y^r \left\{ (\partial_j \Gamma^r_{i\ell}) |m H^s_{rhe} + (\partial_j \Gamma^r_{i\ell}) |m H^s_{rhe} + (\partial_j \Gamma^r_{i\ell}) |m H^s_{rhe} \right\} = 0.
\end{aligned}
\end{equation}

Thus, we conclude

\textbf{Theorem 2.7.} In $K^h - GBR - F_n$, we have the identities (2.43) and (2.44) [provided Cartan fourth curvature tensor $K^i_{jkh}$ is recurrent].

We know that the associate tensor $R^i_{jkh}$ of Cartan’s third curvature tensor $R^i_{jkh}$ satisfies the identity [7]

\begin{equation}
\begin{aligned}
R^i_{jkh} + R^i_{ikh} + R^i_{ihk} + (C_{ijks} K^s_{rkh} + C_{ihks} K^s_{rkh} + C_{ihks} K^s_{rkh}) y^r = 0.
\end{aligned}
\end{equation}

Using (1.11) in (2.45), we get

\begin{equation}
\begin{aligned}
R^i_{jkh} + R^i_{ikh} + R^i_{ihk} + (C_{ijks} H^s_{rkh} + C_{ihks} H^s_{rkh} + C_{ihks} H^s_{rkh}) = 0.
\end{aligned}
\end{equation}

Differentiating (2.46) covariantly with respect to $x^k$ in the sense of Cartan, we get

\begin{equation}
\begin{aligned}
R^i_{jkh} + R^i_{ikh} + R^i_{ihk} + (C_{ijks} H^s_{rkh} + C_{ihks} H^s_{rkh} + C_{ihks} H^s_{rkh}) = 0.
\end{aligned}
\end{equation}

The associate tensor $K_{jikh}$ of Cartan’s fourth curvature tensor $K^i_{jkh}$ and the associate tensor $R^i_{jkh}$ of Cartan’s third curvature tensor $R^i_{jkh}$ are connected by the identity [7]

\begin{equation}
\begin{aligned}
K^i_{jkh} - K^i_{jkh} = 2R^i_{jkh}.
\end{aligned}
\end{equation}

Differentiating (2.48) covariantly with respect to $x^k$ in the sense of Cartan, we get

\begin{equation}
\begin{aligned}
K^i_{jkh} - K^i_{jkh} = 2R^i_{jkh}.
\end{aligned}
\end{equation}

Differentiating (2.49) covariantly with respect to $x^m$ in the sense of Cartan and using (2.2), we get

\begin{equation}
\begin{aligned}
\lambda_s (K^s_{jkh} |m - K^s_{jkh} |m) + b^m (K^s_{jkh} - K^s_{jkh}) = 2R^s_{jkh} |m.
\end{aligned}
\end{equation}

Putting (2.48) and (2.49) in (2.50), we get

\begin{equation}
\begin{aligned}
R^i_{jkh} |m = \lambda_s R^s_{jkh} |m + b^m R^s_{jkh}.
\end{aligned}
\end{equation}

Differentiating (2.51) covariantly with respect to $x^m$ in the sense of Cartan and using (2.51), we get

\begin{equation}
\begin{aligned}
\lambda_s (R^s_{jkh} |m + R^s_{ikh} |m + R^s_{ihk} |m) + b^m (R^s_{jkh} + R^s_{ikh} + R^s_{ikh}) \\
+ (C_{ijks} H^s_{rkh} + C_{ihks} H^s_{rkh} + C_{ihks} H^s_{rkh}) |m = 0.
\end{aligned}
\end{equation}

In view of (2.46) and putting (2.45) in (2.52), we get

\begin{equation}
\begin{aligned}
(C_{ijks} H^s_{rkh} + C_{ihks} H^s_{rkh} + C_{ihks} H^s_{rkh}) |m = \lambda_s (C_{ijks} H^s_{rkh} + C_{ihks} H^s_{rkh} + C_{ihks} H^s_{rkh}) |m \\
+ b^m (C_{ijks} H^s_{rkh} + C_{ihks} H^s_{rkh} + C_{ihks} H^s_{rkh}) |m.
\end{aligned}
\end{equation}

Transvecting (2.53) by $y^j$, using (1.1a), (1.2) and (1.6), we get

\begin{equation}
\begin{aligned}
(C_{ijks} H^s_{rkh} - C_{ihks} H^s_{rkh}) |m = \lambda_s (C_{ijks} H^s_{rkh} - C_{ihks} H^s_{rkh}) |m + b^m (C_{ijks} H^s_{rkh} - C_{ihks} H^s_{rkh}) |m.
\end{aligned}
\end{equation}

Transvecting (2.54) by $g^p$, using (1.1c) and (1.4), we get

\begin{equation}
\begin{aligned}
(C^p_{ks} H^s_{rkh} - C^p_{ks} H^s_{rkh}) |m = \lambda_s (C^p_{ks} H^s_{rkh} - C^p_{ks} H^s_{rkh}) |m + b^m (C^p_{ks} H^s_{rkh} - C^p_{ks} H^s_{rkh}) |m.
\end{aligned}
\end{equation}

Thus, we conclude...
Theorem 2.8. In $K^n$–GBR–$F_n$, the tensors $(C_{ijs} H^s_{ik} + C_{iks} H^s_{jh} + C_{ith} H^s_{kj})$, $(C_{iks} H^s_{ih} - C_{ith} H^s_{ij})$, and $(C_{ks} H^s_{ip} - C_{kps} H^s_{ip})$ are all generalized birecurrent.

References