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Certain Generalized Birecurrent Tensors In $\mathbf{K^h - GBR\text{–} F_n}$.

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Abstract: *We presented a Finsler space* F_n *whose Cartan's fourth curvature tensor* K^i_{jkh} *satisfies* $K^i_{jkh|\ell|m} = \lambda_\ell K^i_{jkh|m} + b_{\ell m} K^i_{jkh}$, $K^i_{jkh} \neq 0$, *where* λ_ℓ *and* $b_{\ell m}$ *are non-zero covariant vector field and covariant tensor field of second order, respectively. such space is called as* K^h*–generalized birecurrent space and denoted briefly by* K^h*–* GBR*–* F_n*. In the present paper we shall obtain some generalized birecurrent tensor in an* $K^h - GBR - F_n$*.*
Keywords: *Finsler space,* K^h*– Generalized birecurrent Finsler space, Ricci tensor.*

I. Introduction

Let F_n be An n-dimensional Finsler space equipped with the metric function a $F(x, y)$ satisfying the request conditions [7].

Cartan's second kind covariant differentiation form arbitrary vector field x^i with respect to x^k is given by [3],[4]

$$X^i_{|k} := \partial_k X^i - \left(\dot{\partial}_r X^i\right) G^r_k + X^r \Gamma^{*i}_{rk} .$$

M. Motsumoto [5],[6] calls this derivative as $h -$ covariant derivative.

The vector y^i and the metric tensor g_{ij} and its associate satisfies the following relations

(1.1) $\quad a)\ \ y^i_{\ |k} = 0 \quad$, b) $g_{ij|k} = 0 \quad$ and c) $g^{ij}_{\ |k}=0$.

The tensor C_{ijk} is known as $(h)hv$ - torsion tensor [5], it is positively homogeneous of degree -1 in y^i and symmetric in all its indices. By using Euler's theorem on homogeneous properties, this tensor satisfies the following

(1.2) $\quad C_{ijk}\, y^i = C_{kij}\, y^i = C_{jki}\, y^i = 0.$

Also satisfies the following relation

(1.3) $\quad C_{ijk}\, g^{jk} = C_i.$

The $(v)hv$-torsion tensor C^i_{jk} is the associate tensor of the $(h)hv$-tensor C_{ijk} and defined by

(1.4) $\quad C^h_{ik} := g^{hj} C_{ijk}$.

The tensor C^h_{ik} is positively homogeneous of degree -1 in y^i and symmetric in its lower indices.

The tensor $P^{\ i}_{jk}$ is called the $v(hv)$-torsion tensor and given by

(1.5) $\quad P^r_{jk} = \left(\dot{\partial}_j \Gamma^{*r}_{hk}\right) y^h = \Gamma^{*r}_{jhk} y^h$.

Berwald curvature tensor H^i_{jkh} and the $h(v)$- torsion tensor $H^{\ i}_{kh}$ are related by

$$H^{\ i}_{jkh}\, y^j = H^{\ i}_{kh} .$$

The deviation tensor $H^{\ i}_k$ is positively homogeneous of degree two in y^i and satisfies

(1.6) $\quad H^{\ i}_{hk}\, y^h = H^{\ i}_k$.

Cartan's fourth curvature tensor $K^{\ i}_{jkh}$ satisfies the following identity known as *Bianchi identity*

(1.7) $K^{\ i}_{jkh|\ell} + K^{\ i}_{j\ell k|h} + K^{\ i}_{jh\ell|k} + y^r\{(\dot{\partial}_s \Gamma^{*i}_{jk}) K^{\ s}_{rh\ell} + (\dot{\partial}_s \Gamma^{*i}_{j\ell}) K^s_{rkh} + (\dot{\partial}_s \Gamma^{*i}_{jh}) K^{\ s}_{r\ell k}\} = 0.$

The associate tensor K_{ijkh} of the curvature tensor $K^{\ i}_{jkh}$ is given by

(1.8) $\quad K_{ijkh} := g_{rj} K^{\ r}_{ikh}$.

The tensor K_{ijkh} also satisfies the condition

(1.9) $\quad K_{hijk} + K_{ihjk} = -2\, C_{hir}\, K^{\ r}_{sjk}\, y^s.$

The curvature tensor $K^{\ i}_{jkh}$ satisfies the following relations too

(1.10) $\quad K^{\ i}_{jkh} y^j = H^{\ i}_{kh}$, (1.11) $K^{i}_{jki} = K_{jk}$

and

(1.12) $H^{i}_{jkh} - K^{i}_{jkh} = P^{i}_{jk|h} + P^{r}_{jk} P^{i}_{rh} - h/k^{*}$.

$* - h/k$ means the subtraction from the former term by interchange the indices h and k.

N. S. H. Hussien [4] and M. A. A. Ali [1] obtained some birecurrent tensors in a K^{h} – birecurrent Finsler space.

II. Certain Generalized Birecurrent Tensors

Let us consider an $K^{h}-GBR-F_{n}$ characterized by the condition

(2.1) $K^{i}_{jkh|\ell|m} = \lambda_{\ell} K^{i}_{jkh|m} + b_{\ell m} K^{i}_{jkh}$, $K^{i}_{jkh} \neq 0$

where λ_{ℓ} and $b_{\ell m} = \lambda_{\ell|m}$ are non-zero covariant vector fields and covariant tensor field of second order, respectively. The space and the tensor satisfying the condition (2.1) will be called K^{h}*–generalized birecurrent space* and h*–generalized birecurrent tensor,* respectively. We shall denote them briefly by $K^{h}-GBR-F_{n}$ and $h-GBR$, respectively.

Transvecting (2.1) by the metric tensor g_{ip} , using (1.8) and (1.1b), we get

(2.2) $K_{jpkh|\ell|m} = \lambda_{\ell} K_{jpkh|m} + b_{\ell m} K_{jpkh}$.

Contracting the indices i and h in (2.2) and using (1.11), we get

(2.3) $K_{jk|\ell|m} = \lambda_{\ell} K_{jk|m} + b_{\ell m} K_{jk}$.

Transvecting (2.3) by y^{k} and using (1.1a), we get

(2.4) $K_{j|\ell|m} = \lambda_{\ell} K_{j|m} + b_{\ell m} K_{j}$.

where $K_{jk} y^{k} = K_{j}$.

Transvecting (2.1) by y^{j} , using (1.1a) and (1.10), we get

(2.5) $H^{i}_{kh|\ell|m} = \lambda_{\ell} H^{i}_{kh|m} + b_{\ell m} H^{i}_{kh}$.

Contracting the indices i and h in (2.5) and using ($H_{k} = H^{i}_{ki}$), we get

(2.6) $H_{k|\ell|m} = \lambda_{\ell} H_{k|m} + b_{\ell m} H_{k}$.

Differentiating (1.9) covariantly with respect to x^{ℓ} in the sense of Cartan and using(1.10), we get

(2.7) $K_{hijk|\ell} + K_{ihjk|\ell} = (-2C_{hir} H^{r}_{jk})_{|\ell}$.

Differentiating (2.7) covariantly with respect to x^{m} in the sense of Cartan, we get

(2.8) $K_{hijk|\ell|m} + K_{ihjk|\ell|m} = (-2C_{hir} H^{r}_{jk})_{|\ell|m}$.

Using (2.2) in (2.8), we get

(2.9) $\lambda_{\ell}(K_{hijk|m} + K_{ihjk|m}) + b_{\ell m}(K_{hijk} + K_{ihjk}) = (-2C_{hir} H^{r}_{jk})_{|\ell|m}$.

Putting (1.9), (1.10) and (2.7) in (2.9), we get

(2.10) $(C_{hir} H^{r}_{jk})_{|\ell|m} = \lambda_{\ell}(C_{hir} H^{r}_{jk})_{|m} + b_{\ell m}(C_{hir} H^{r}_{jk})$.

Transvecting (2.10) by g^{hp}, using (1.1c) and (1.4), we get

(2.11) $(C^{p}_{ir} H^{r}_{jk})_{|\ell|m} = \lambda_{\ell}(C^{p}_{ir} H^{r}_{jk})_{|m} + b_{\ell m}(C^{p}_{ir} H^{r}_{jk})$.

Transvecting (2.11) by y^{j}, using (1.1a) and (1.6), we get

(2.12) $(C^{p}_{ir} H^{r}_{k})_{|\ell|m} = \lambda_{\ell}(C^{p}_{ir} H^{r}_{k})_{|m} + b_{\ell m}(C^{p}_{ir} H^{r}_{k})$.

Transvecting (2.10) by g^{hi}, using (1.1c) and (1.3), we get

(2.13) $(C_{r} H^{r}_{jk})_{|\ell|m} = \lambda_{\ell}(C_{r} H^{r}_{jk})_{|m} + b_{\ell m}(C_{r} H^{r}_{jk})$.

Transvecting (2.13) by y^{j}, using (1.1a) and (1.6), we get

(2.14) $(C_{r} H^{r}_{k})_{|\ell|m} = \lambda_{\ell}(C_{r} H^{r}_{k})_{|m} + b_{\ell m}(C_{r} H^{r}_{k})$.

Contracting the indices p and k in (2.12), we get

(2.15) $(C^{p}_{ir} H^{r}_{p})_{|\ell|m} = \lambda_{\ell}(C^{p}_{ir} H^{r}_{p})_{|m} + b_{\ell m}(C^{p}_{ir} H^{r}_{p})$.

Thus, we conclude

Theorem 2.1. *In $K^{h}-GBR-F_{n}$, the tensors $(C_{hir} H^{r}_{jk})$,$(C^{p}_{ir} H^{r}_{jk})$,$(C^{p}_{ir} H^{r}_{k})$, $(C_{r} H^{r}_{jk})$, $(C_{r} H^{r}_{k})$ and $(C^{p}_{ir} H^{r}_{p})$ are all generalized birecurrent.*

We know the identity [7]

(2.16) $K_j = H_j - H_j^i C_i$.

Differentiating (2.16) covariantly with respect to x^ℓ in the sense of Cartan, we get

(2.17) $K_{j|\ell} = H_{j|\ell} - (H_j^i C_i)_{|\ell}$.

Differentiating (2.17) covariantly with respect to x^m in the sense of Cartan, we get

(2.18) $K_{j|\ell|m} = H_{j|\ell|m} - (H_j^i C_i)_{|\ell|m}$.

Using (2.4) and (2.14) in (2.18), we get

(2.19) $K_{j|\ell|m} = \lambda_\ell \{H_{j|m} - (H_j^i C_i)_{|m}\} + b_{\ell m}\{H_j - (H_j^i C_i)\}$.

Putting (2.16) and (2.17) in (2.19), we get

(2.20) $K_{j|\ell|m} = \lambda_\ell K_{j|m} + b_{\ell m} K_j$.

Thus, we conclude

Theorem 2.2. *In $K^h - GBR - F_n$, the vector K_j is generalized birecurrent.*

Also, we have the identity [7]

(2.21) $R_j = K_j + C_{jr}^i H_i^r$.

Differentiating (2.21) covariantly with respect to x^ℓ in the sense of Cartan, we get

(2.22) $R_{j|\ell} = K_{j|\ell} + (C_{jr}^i H_i^r)_{|\ell}$.

Differentiating (2.22) covariantly with respect to x^m in the sense of Cartan, we get

(2.23) $R_{j|\ell|m} = K_{j|\ell|m} + (C_{jr}^i H_i^r)_{|\ell|m}$.

Using (2.15) and (2.20) in (2.23), we get

(2.24) $R_{j|\ell|m} = \lambda_\ell \{K_{j|m} + (C_{jr}^i H_i^r)_{|m}\} + b_{\ell m}\{K_j + (C_{jr}^i H_i^r)\}$.

Putting (2.21) and (2.22) in (2.24), we get

$$R_{j|\ell|m} = \lambda_l R_{j|m} + b_{\ell m} R_j .$$

Thus, we conclude

Theorem 2.3. *In $K^h - GBR - F_n$, the vector R_j is generalized birecurrent.*

We have Cartan's fourth curvature tensor K_{jkh}^i, $v(hv)-$ torsion tensor P_{jk}^i and Berwald curvature tensor H_{jkh}^i are connected by the formula (1.12).

Differentiating (1.12) covariantly with respect to x^ℓ in the sense of Cartan, we get

(2.25) $H_{jkh|\ell}^i - K_{jkh|\ell}^i = (P_{jk|h}^i + P_{jk}^r P_{rh}^i - h/k)_{|\ell}$.

Differentiating (2.25) covariantly with respect to x^m in the sense of Cartan, we get

(2.26) $H_{jkh|\ell|m}^i - K_{jkh|\ell|m}^i = (P_{jk|h}^i + P_{jk}^r P_{rh}^i - h/k)_{|\ell|m}$.

Using (2.1) and if Berwald curvature tensor H_{jkh}^i is generalized birecurrent, (2.26) reduces to

(2.27) $\lambda_\ell (H_{jkh|m}^i - K_{jkh|m}^i) + b_{\ell m}(H_{jkh}^i - K_{jkh}^i) = (P_{jk|h}^i + P_{jk}^r P_{rh}^i - h/k)_{|\ell|m}$.

Putting (1.12) and (2.25) in (2.27), we get

$$(P_{jk|h}^i + P_{jk}^r P_{rh}^i - h/k)_{|\ell|m} = \lambda_\ell (P_{jk|h}^i + P_{jk}^r P_{rh}^i - h/k)_{|m}$$
$$+ b_{\ell m}(P_{jk|h}^i + P_{jk}^r P_{rh}^i - h/k).$$

Thus, we conclude

Theorem 2.4. *In $K^h - GBR - F_n$, the tensor $(P_{jk|h}^i + P_{jk}^r P_{rh}^i - h/k)$ is generalized* birecurrent [provided Berwald curvature tensor H_{jkh}^i is generalized birecurrent].

We know the curvature tensor K_{ijkh} satisfies [5] the identity

(2.28) $K_{hijk} - K_{jkhi} = H_{hj}^r C_{rik} - H_{hk}^r C_{rij} + H_{ik}^r C_{rhj} - H_{ij}^r C_{rhk} - H_{jk}^r C_{rhi} + H_{hi}^r C_{rjk}$.

Differentiating (2.28) covariantly with respect to x^ℓ in the sense of Cartan, we get

(2.29) $K_{hijk|\ell} - K_{jkhi|\ell} = (H_{hj}^r C_{rik} - H_{hk}^r C_{rij} + H_{ik}^r C_{rhj} - H_{ij}^r C_{rhk}$
$- H_{jk}^r C_{rhi} + H_{hi}^r C_{rjk})_{|\ell}$.

Differentiating (2.29) covariantly with respect to x^m in the sense of Cartan, we get

(2.30) $K_{hijk|\ell|m} - K_{jkhi|\ell|m} = (H_{hj}^r C_{rik} - H_{hk}^r C_{rij} + H_{ik}^r C_{rhj} - H_{ij}^r C_{rhk}$

$$- H^r_{jk}\, C_{rhi} + H^r_{hi}\, C_{rjk}\,)_{|\ell|m}.$$

Using (2.2) in (2.30), we get

(2.31) $\lambda_\ell\big(K_{hijk\,|\ell} - K_{jk\,hi|\ell}\big) + b_{\ell m}\big(K_{hijk} - K_{jk\,hi}\big) = (H^r_{hj}\, C_{rik} - H^r_{hk}\, C_{rij} +$
$H^r_{ik}\, C_{rhj} - H^r_{ij}\, C_{rhk} - H^r_{jk}\, C_{rhi} + H^r_{hi}\, C_{rjk}\,)_{|\ell|m}.$

Putting (2.28) and (2.29) in (2.31), we get

(2.32) $(\,H^r_{hj}\, C_{rik} - H^r_{hk}\, C_{rij} + H^r_{ik}\, C_{rhj} - H^r_{ij}\, C_{rhk} - H^r_{jk}\, C_{rhi} + H^r_{hi}\, C_{rjk})_{|\ell|m}$
$= \lambda_\ell(\,H^r_{hj}\, C_{rik} - H^r_{hk}\, C_{rij} + H^r_{ik}\, C_{rhj} - H^r_{ij}\, C_{rhk} - H^r_{jk}\, C_{rhi} + H^r_{hi}\, C_{rjk})_{|m}$
$+ b_{\ell m}\big(H^r_{hj}\, C_{rik} - H^r_{hk}\, C_{rij} + H^r_{ik}\, C_{rhj} - H^r_{ij}\, C_{rhk} - H^r_{jk}\, C_{rhi} + H^r_{hi}\, C_{rjk}\big)$

Transvecting (2.32) by y^j , using (1.1a), (1.2) and (1.6), we get

(2.33) $(H^r_h\, C_{rik} - H^r_i\, C_{rhk} + H^r_k\, C_{rhi})_{|\ell|m} = \lambda_\ell(H^r_h\, C_{rik} - H^r_i\, C_{rhk} + H^r_k\, C_{rhi})_{|m}$
$+b_{\ell m}\, (H^r_h\, C_{rik} - H^r_i\, C_{rhk} + H^r_k\, C_{rhi}).$

Transvecting (2.33) by g^{pr} , using (1.1c) and (1.4), we get

$$(H^r_h\, C^p_{ik} - H^r_i\, C^p_{hk} + H^r_k\, C^p_{hi})_{|\ell|m} = \lambda_\ell(H^r_h\, C^p_{ik} - H^r_i\, C^p_{hk} + H^r_k\, C^p_{hi})_{|m}$$
$$+b_{\ell m}(H^r_h\, C^p_{ik} - H^r_i\, C^p_{hk} + H^r_k\, C^p_{hi}).$$

Thus, we conclude

Theorem 2.5. *In $K^h - GBR - F_n$, the tensors $\big(H^r_{hj}\, C_{rik} - H^r_{hk} C_{rij} + H^r_{ik}\, C_{rhj} - H^r_{ij}\, C_{rhk} - H^r_{jk}\, C_{rhi} +$ Hhi rCrjk, Hh rCrik−Hi rCrhk+Hk rCrhi and (Hh rCikp− Hi rChkp+ Hk rChip) are all generalized birecurrent.*

We have the identity [7]

(2.34) $K_{ij\,hk} + K_{ikj\,h} + K_{ihkj} = -2\, y^r\big(\, C_{ijs}\, K^s_{rhk} + C_{iks}\, K^s_{rj\,h} + C_{ihs}\, K^s_{rkj}\, \big).$

Using (1.10) in (2.34), we get

(2.35) $K_{ij\,hk} + K_{ikj\,h} + K_{ihkj} = -2\,\big(C_{ijs}\, H^s_{hk} + C_{iks}\, H^s_{jh} + C_{ihs}\, H^s_{kj}\,\big).$

Differentiating (2.35) covariantly with respect to x^ℓ in the sense of Cartan, we get

(2.36) $K_{ij\,hk|\ell} + K_{ikj\,h|\ell} + K_{ihkj\,|\ell} = -2\big(C_{ijs} H^s_{hk} + C_{iks} H^s_{jh} + C_{ihs} H^s_{kj}\,\big)_{|\ell}.$

Differentiating (2.36) covariantly with respect to x^m in the sense of Cartan, we get

(2.37) $K_{ij\,hk|\ell|m} + K_{ikj\,h|\ell|m} + K_{ihkj\,|\ell|m} = -2\big(C_{ijs} H^s_{hk} + C_{iks} H^s_{jh} + C_{ihs} H^s_{kj}\big)_{|\ell|m}.$

Using (2.2), (2.35) and (2.36) in (2.37), we get

(2.38) $\big(C_{ijs} H^s_{hk} + C_{iks} H^s_{jh} + C_{ihs} H^s_{kj}\,\big)_{|\ell|m} = \lambda_\ell\big(C_{ijs} H^s_{hk} + C_{iks} H^s_{jh} + C_{ihs} H^s_{kj}\big)_{|m}$
$+b_{\ell m}\big(C_{ijs} H^s_{hk} + C_{iks} H^s_{jh} + C_{ihs} H^s_{kj}\,\big).$

Transvecting (2.38) by y^j, using (1.1a), (1.2) and (1.6), we get

(2.39) $\big(C_{iks} H^s_h - C_{ihs} H^s_k\,\big)_{|\ell|m} = \lambda_\ell\big(C_{iks}\, H^s_h - C_{ihs} H^s_k\,\big)_{|m} + b_{\ell m}\big(C_{iks} H^s_h - C_{ihs} H^s_k\,\big).$

Transvecting (2.39) by g^{pr}, using (1.1c) and (1.4), we get

$$\big(C^p_{ks} H^s_h - C^p_{hs} H^s_k\,\big)_{|\ell|m} = \lambda_\ell\big(C^p_{ks}\, H^s_h - C^p_{hs}\, H^s_k\,\big)_{|m} + b_{\ell m}\big(C^p_{ks} H^s_h - C^p_{hs} H^s_k\,\big).$$

Thus, we conclude

Theorem 2.6. *In $K^h - GBR - F_n$, the tensors $\big(C_{ijs}\, H^s_{hk} + C_{iks}\, H^s_{jh} + C_{ihs}\, H^s_{kj}\,\big)$, $(\,C_{iks}\, H^s_h - C_{ihs}\, H^s_k\,)$ and $\big(\,C^p_{ks}\, H^s_h - C^p_{hs}\, H^s_k\,\big)$ are all generalized birecurrent.*

Differentiating (1.7) covariantly with respect to x^m in the sense of Cartan, we get

(2.40) $K^i_{jk\,h|\ell|m} + K^i_{j\ell k|h|m} + K^i_{jh\ell|k|m} + y^r\big\{\big(\dot\partial_s \Gamma^{*i}_{jk}\big)K^s_{rh\ell|m} + \big(\dot\partial_s \Gamma^{*i}_{j\ell}\big)K^s_{rk\,h|m}$
$+\big(\dot\partial_s \Gamma^{*i}_{jh}\big)K^s_{r\ell k|m}\big\} + y^r\,\Big\{\big(\dot\partial_s \Gamma^{*i}_{jk}\big)_{|m} K^s_{rh\ell} + \big(\dot\partial_s \Gamma^{*i}_{j\ell}\big)_{|m} K^s_{rk\,h}$
$+\big(\dot\partial_s \Gamma^{*i}_{jh}\big)_{|m} K^s_{r\ell k}\Big\} = 0.$

Using (2.1) in (2.40), we get

(2.41) $\lambda_\ell K^i_{jk\,h|m} + \lambda_h K^i_{j\ell k|m} + \lambda_k\, K^i_{jh\ell|m} + b_{\ell m}\, K^i_{jk\,h} + b_{hm} K^i_{j\ell k} + b_{km} K^i_{jh\ell}$
$+y^r\,\big\{\big(\dot\partial_s \Gamma^{*i}_{jk}\big)K^s_{rh\ell|m} + \big(\dot\partial_s \Gamma^{*i}_{j\ell}\big)K^s_{rk\,h|m} + \big(\dot\partial_s \Gamma^{*i}_{jh}\big)K^s_{r\ell k|m}\big\}$
$+y^r\,\Big\{\big(\dot\partial_s \Gamma^{*i}_{jk}\big)_{|m} K^s_{rh\ell} + \big(\dot\partial_s \Gamma^{*i}_{j\ell}\big)_{|m} K^s_{rk\,h} + \big(\dot\partial_s \Gamma^{*i}_{jh}\big)_{|m} K^s_{r\ell k}\Big\} = 0.$

If Cartan's fourth curvature tensor K^i_{jkh} is recurrent, (2.41) becomes

(2.42) $\lambda_\ell \lambda_m K^i_{jkh} + \lambda_h \lambda_m K^i_{j\ell k} + \lambda_k \lambda_m K^i_{jh\ell} + b_{\ell m} K^i_{jkh} + b_{hm} K^i_{j\ell k} + b_{km} K^i_{jh\ell}$
$+ \lambda_m y^r\{(\dot\partial_s \Gamma^{*i}_{jk}) K^s_{rh\ell} + (\dot\partial_s \Gamma^{*i}_{j\ell}) K^s_{rkh} + (\dot\partial_s \Gamma^{*i}_{jh}) K^s_{r\ell k}\}$
$+ y^r \left\{(\dot\partial_s \Gamma^{*i}_{jk})_{|m} K^s_{rh\ell} + (\dot\partial_s \Gamma^{*i}_{j\ell})_{|m} K^s_{rkh} + (\dot\partial_s \Gamma^{*i}_{jh})_{|m} K^s_{r\ell k}\right\} = 0$.

Putting (1.7) in (2.42), we get

$$\lambda_\ell \lambda_m K^i_{jkh} + \lambda_h \lambda_m K^i_{j\ell k} + \lambda_k \lambda_m K^i_{jh\ell} + b_{\ell m} K^i_{jkh} + b_{hm} K^i_{j\ell k} + b_{km} K^i_{jh\ell}$$
$$-\lambda_m(K^i_{jkh|\ell} + K^i_{j\ell k|h} + K^i_{jh\ell|k}) + y^r \left\{(\dot\partial_s \Gamma^{*i}_{jk})_{|m} K^s_{rh\ell} + (\dot\partial_s \Gamma^{*i}_{j\ell})_{|m} K^s_{rkh} +\right.$$
$$\left.(\dot\partial_s \Gamma^{*i}_{jh})_{|m} K^s_{r\ell k}\right\} = 0.$$

which can be written as

(2.43) $b_{\ell m} K^i_{jkh} + b_{hm} K^i_{j\ell k} + b_{km} K^i_{jh\ell} + + y^r \left\{(\dot\partial_s \Gamma^{*i}_{jk})_{|m} K^s_{rh\ell} +\right.$
$\left.(\dot\partial_s \Gamma^{*i}_{j\ell})_{|m} K^s_{rkh} + (\dot\partial_s \Gamma^{*i}_{jh})_{|m} K^s_{r\ell k}\right\} = 0$

Transvecting (2.43) by y^j, using (1.1a), (1.10) and (1.5), we get

(2.44) $b_{\ell m} H^i_{kh} + b_{hm} H^i_{\ell k} + b_{km} H^i_{h\ell} + P^i_{sk|m} H^s_{h\ell} + P^i_{s\ell|m} H^s_{kh} + P^i_{sh|m} H^s_{\ell k} = 0$.

Thus, we conclude

Theorem 2.7. *In* $K^h - GBR - F_n$, *we have the identities* (2.43) *and* (2.44) [provided Cartan fourth curvature tensor K^i_{jkh} is recurrent].

We know that the associate tensor R_{ijkh} of Cartan's third curvature tensor R^i_{jkh} satisfies the identity [7]

(2.45) $R_{ijhk} + R_{ikjh} + R_{ihkj} + (C_{ijs} K^s_{rhk} + C_{iks} K^s_{rjh} + C_{ihs} K^s_{rkj})y^r = 0.$

Using (1.11) in (2.45), we get

(2.46) $R_{ijhk} + R_{ikjh} + R_{ihkj} + C_{ijs} H^s_{hk} + C_{iks} H^s_{jh} + C_{ihs} H^s_{kj} = 0.$

Differentiating (2.46) covariantly with respect to x^ℓ in the sense of Cartan, we get

(2.47) $R_{ijhk|\ell} + R_{ikjh|\ell} + R_{ihkj|\ell} + (C_{ijs} H^s_{hk} + C_{iks} H^s_{jh} + C_{ihs} H^s_{kj})_{|\ell} = 0.$

The associate tensor K_{ijkh} of Cartan's fourth curvature tensor K^i_{jkh} and the associate tensor R_{ijkh} of Cartan's third curvature tensor R^i_{jkh} are connected by the identity [7]

(2.48) $K_{hijk} - K_{ihjk} = 2R_{hijk}$.

Differentiating (2.48) covariantly with respect to x^ℓ in the sense of Cartan, we get

(2.49) $K_{hijk|\ell} - K_{ihjk|\ell} = 2R_{hijk|\ell}$.

Differentiating (2.49) covariantly with respect to x^m in the sense of Cartan and using (2.2), we get

(2.50) $\lambda_\ell(K_{hijk|m} - K_{ihjk|m}) + b_{\ell m}(K_{hijk} - K_{ihjk}) = 2R_{hijk|\ell|m}$.

Putting (2.48) and (2.49) in (2. 50), we get

(2.51) $R_{hijk|\ell|m} = \lambda_\ell R_{hijk|m} + b_{\ell m} R_{hijk}$.

Differentiating (2.47) covariantly with respect to x^m in the sense of Cartan and using (2.51), we get

(2.52) $\lambda_\ell(R_{ijhk|m} + R_{ikjh|m} + R_{ihkj|m}) + b_{\ell m}(R_{ijhk} + R_{ikjh} + R_{ihkj})$
$+(C_{ijs} H^s_{hk} + C_{iks} H^s_{jh} + C_{ihs} H^s_{kj})_{|\ell|m} = 0.$

In view of (2.46) and putting (2.45) in (2.52), we get

(2.53) $(C_{ijs} H^s_{hk} + C_{iks} H^s_{jh} + C_{ihs} H^s_{kj})_{|\ell|m} = \lambda_\ell(C_{ijs} H^s_{hk} + C_{iks} H^s_{jh} + C_{ihs} H^s_{kj})_{|m}$
$+ b_{\ell m}(C_{ijs} H^s_{hk} + C_{iks} H^s_{jh} + C_{ihs} H^s_{kj})$.

Transvecting (2.53) by y^j, using (1.1a), (1.2) and (1.6), we get

(2.54) $(C_{iks} H^s_h - C_{ihs} H^s_k)_{|l|m} = \lambda_\ell(C_{iks} H^s_h - C_{ihs} H^s_k)_{|m} + b_{\ell m}(C_{iks} H^s_h - C_{ihs} H^s_k)$.

Transvecting (2.54) by g^{pi}, using (1.1c) and (1.4), we get

$$(C^p_{ks} H^s_h - C^p_{hs} H^s_k)_{|\ell|m} = \lambda_\ell(C^p_{ks} H^s_h - C^p_{hs} H^s_k)_{|m} + b_{\ell m}(C^p_{ks} H^s_h - C^p_{hs} H^s_k).$$

Thus, we conclude

Theorem 2.8. *In $K^h - GBR - F_n$, the tensors* $\left(C_{ijs}\ H^s_{hk} + C_{iks}\ H^s_{jh} + C_{ihs}\ H^s_{kj}\right)$, $\left(C_{iks}\ H^s_h - C_{ihs}\ H^s_k\right)$ *and* $\left(C^p_{ks}\ H^s_h - C^p_{hs}\ H^s_k\right)$ *are all generalized birecurrent.*

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