International Journal of Mathematics and Statistics Invention (IJMSI)
E-ISSN: 2321 – 4767 P-ISSN: 2321 - 4759
www.ijmsi.org Volume 5 Issue 2 || February. 2017 || PP-37-40

Lines Parallel to One Side of Triangle Related to Basic Means

A. Altıntaş, L.Giugiuc, Z.Akça
[1] *(Emirdağ Anadolu Lisesi, Afyonkarahisar,Turkey)*
[5]*(National College, Traian- Drobeta Turnu Severin/ Romania)*
(Department of Mathematics Computer Sciences, Eskişehir Osmangazi University,26480,Eskişehir,Turkey)

ABSTRACT: *In this work we illustrate the lines which are parallel to one side of triangle related to basic means using homogenous barycentric coordinates of a triangle.*
KEYWORDS: *barycentric coordinates, means, geometry*

I. INTRODUCTION

The notion of barycentric coordinates dates back to Mobius. In a given triangle ABC, every point P is coordinatized by a triple of numbers $(u : v : w)$ in such a way that the system of masses u at A, v at B, and w at C will have its balance point at P. These masses can be taken in the proportions of the areas of triangle PBC, PCA and PAB. Allowing the point P to be outside the triangle, we use signed areas of oriented triangles [1]. The homogeneous barycentric coordinates of P with reference to ABC is a triple of numbers $(x : y : z)$ such that;

$$x : y : z = \Delta PBC : \Delta PCA : \Delta PAB$$

The three vertices of a triangle have barycentric coordinates $A(1:0:0)$, $B(0:1:0)$ and $C(0:0:1)$ .The centroid has coordinates $G = (1:1:1)$ since areas of triangles $\Delta PBC, \Delta PCA, \Delta PAB$ are equal. Since barycentric coordinates are homogenous the point $(kx : ky : kz)$ will refer to the point $(x : y : z)$ for $k \neq 0$. Some points and their barycentric coordinates are as follows:

Incenter
$$I = (a : b : c)$$

excenters
$$I_A = (-a : b : c)$$
$$I_B = (a : -b : c)$$
$$I_C = (a : b : -c)$$

Circumcircle center
$$O = (a^2(b^2 + c^2 - a^2) : b^2(c^2 + a^2 - b^2) : c^2(a^2 + b^2 - c^2))$$

Orthocenter
$$H = ((a^2 + b^2 - c^2)(a^2 - b^2 + c^2) : (a^2 + b^2 - c^2)(-a^2 + b^2 + c^2) : (a^2 - b^2 + c^2)(-a^2 + b^2 + c^2))$$

The first Brocard point Ω is the interior point Ω of a triangle ABC with points labeled in counterclockwise order for which the angles $\angle \Omega AB$, $\angle \Omega BC$, and $\angle \Omega CA$ are equal, with the unique such angle denoted Ω and called the brocard angle. First brocard point has barycentric coordinates $\Omega = \left(\frac{1}{b^2} : \frac{1}{c^2} : \frac{1}{a^2} \right)$. (Fig. 1)

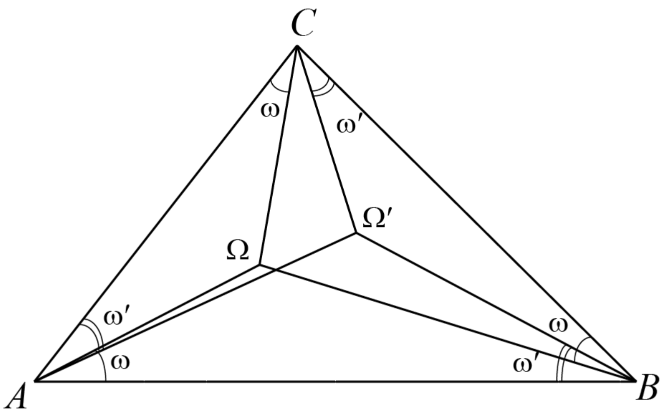

Figure 1. Brocard Points of ABC

The isogonal conjugate of the point $P = (x : y : z)$ is

$$P^* = \left(\frac{a^2}{x} : \frac{b^2}{y} : \frac{c^2}{z} \right)$$

The symmedian point of a triangle is defined as the isogonal conjugate of the centroid of the triangle. Since G has coordinates $(1:1:1)$, symmedian point of triangle has coordinates $K = (a^2 : b^2 : c^2)$. First and second brocard points are isogonal conjugate points. Since $\Omega = \left(\frac{1}{b^2} : \frac{1}{c^2} : \frac{1}{a^2} \right)$, barycentric coodinate of second brocard point is $\Omega' = \left(\frac{1}{c^2} : \frac{1}{a^2} : \frac{1}{b^2} \right)$.

Two points P and Q (not on any of the side lines of the reference triangle) are said to be isotomic conjugates if their respective traces are symmetric with respect to the midpoints of the corresponding sides. We shall denote the isotomic conjugate of P by $P^{\bullet}$ [2]. If $P = (x : y : z)$ then,

$$P^{\bullet} = \left(\frac{1}{x} : \frac{1}{y} : \frac{1}{z} \right)$$

In order to get the equation of a parallel line we consider the infinite points. We know that each line has an infinite point and all infinite points lie on a line called line at infinity. The line at infinity has equation $x + y + z = 0$. The infinite point of the line $px + qy + rz = 0$ is $(q - r : r - p : p - q)$. The infinite points of the side lines BC, CA, AB are $(0 : -1 : 1), (1 : 0 : -1), (-1 : 1, 0)$ respectively [3].

The points $P = (x_1 : y_1 : z_1)$, $Q = (x_2 : y_2 : z_2)$ and $R = (x_3 : y_3 : z_3)$ are collinear if and only if

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0$$

II. LINES PARALEL TO ONE SIDE OF TRIANGLE RELATED TO BASIC MEANS

It is well known that, some basic means are as follows for two positive numbers;

Arithmetic Mean: $A(a,b) = \frac{a+b}{2}$, Geometri Mean $G(a,b) = \sqrt{ab}$, Harmonic Mean: $H(a,b) = \frac{2}{\frac{1}{a} + \frac{1}{b}}$,

Root mean square: $R(a,b) = \sqrt{\frac{a^2 + b^2}{2}}$.

Proposition 2.1. *Let ABC be triangle whose lengths of the sides are a, b and c. G and K are its centroid and symmedian point respectively. GK is parallel to AB if $c = \sqrt{\frac{a^2 + b^2}{2}}$, which is the root mean square of a and b.*

Proof. G and K have the barycentric coordinates $(1:1:1)$ and $(a^2 : b^2 : c^2)$, respectively.The barycentric coordinate of the ideal point of side AB is $(-1:1:0)$. The line GK is parallel to AB if GK passing through the ideal point of AB. Since the determinant in terms of the barycentric coordinates of the ideal point, $G$and K is zero, the equation $a^2 + b^2 - 2c^2 = 0$ is obtained. So, we reach $c = \sqrt{\frac{a^2 + b^2}{2}}$.

$$\begin{vmatrix} -1 & 1 & 0 \\ 1 & 1 & 1 \\ a^2 & b^2 & c^2 \end{vmatrix} = 0 \Leftrightarrow a^2 + b^2 - 2c^2 = 0 \Leftrightarrow c = \sqrt{\frac{a^2 + b^2}{2}}$$

Proposition 2.2. *Let ABC be triangle whose lengths of the sides are a, b and c. Let I' be the isotomic conjugate of incenter I and G be the centroid of the triangle ABC. The line passing through I' and G is parallel to the edge AB iff the length of AB is the harmonic mean of a and b.*

Proof. In homogenous barycentric coordinates G has coordinate $(1:1:1)$ and since I has coordinate $(a:b:c)$, I' has coordinate $(\frac{1}{a}:\frac{1}{b}:\frac{1}{c})$. The ideal point AB is $(-1:1:0)$. Line $I'G$ is parallel to AB if $I'G$ pass through the infinite point of AB. Since the determinant in terms of the barycentric coordinates of the ideal point, I' nd G is zero, the equation $\frac{1}{a} + \frac{1}{b} - \frac{2}{c} = 0$ is obtained. So, we reach

$$\begin{vmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ a^{-1} & b^{-1} & c^{-1} \end{vmatrix} = 0 \Leftrightarrow \frac{1}{a} + \frac{1}{b} - \frac{2}{c} = 0 \Leftrightarrow c = \frac{2}{\frac{1}{a} + \frac{1}{b}}$$

Proposition 2.3. *Let ABC be triangle whose lengths of the sides are a, b and c. Ω is first Brocard point and K is symmedian point of ABC. ΩK is parallel to BC iff $c = \sqrt{ab}$, which is the geometric mean of a and b.*

Proof. In homogenous barycentric coordinates K has coordinate $(a^2 : b^2 : c^2)$ and since Ω has coordinate $\left(\frac{1}{b^2} : \frac{1}{c^2} : \frac{1}{a^2}\right)$. The ideal point of side BC is $(0:-1:1)$. Line ΩK is parallel to BC if ΩK pass through the ideal point of BC.

$$\begin{vmatrix} 0 & -1 & 1 \\ a^2 & b^2 & c^2 \\ b^{-2} & c^{-2} & a^{-2} \end{vmatrix} = 0 \Leftrightarrow \frac{a^2}{c^2} - \frac{c^2}{b^2} = 0 \Leftrightarrow c = \sqrt{ab}$$

Proposition 2.4. *Let ABC be triangle whose lengths of the sides are a, b and c. I is incenter and G is centroid of ABC. IG is parallel to AB if $c = \frac{a+b}{2}$ which is the arithmetic mean of a and b.*

Proof. In homogenous barycentric coordinates I has coordinate $(a:b:c)$ and G has coordinate

$(1:1:1)$. The ideal point of side AB is $(-1:1:0)$. Line IG (Nagel line of ABC) is parallel to AB iff IG pass through the ideal point of AB.

$$\begin{vmatrix} 0 & -1 & 1 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = 0 \Leftrightarrow 2c - b - a = 0 \Leftrightarrow c = \frac{a+b}{2}$$

Exercises 2.5.

1. Let ABC be triangle whose lengths of the sides are a, b and c. I is incenter and I' is isotomic conjugate of incenter I of ABC. II' is parallel to AB if $a = \sqrt{bc}$, which is the geometric mean of b and c.

2. Let ABC be triangle whose lengths of the sides are a, b and c. Ω' is second brocard point of ABC and G is centroid of ABC. $\Omega'G$ is parallel to BC if $c^2 = \frac{2}{\frac{1}{a^2}+\frac{1}{b^2}}$, which is the harmonic mean of a^2 and b^2.

3. For positive numbers a and b contraharmonic mean is defined as $C(a,b) = \frac{a^2+b^2}{a+b}$. Let ABC be triangle whose lengths of the sides are a, b and c. I is incenter and K is symmedian point of ABC. IK is parallel to AB if $c = \frac{a^2+b^2}{a+b}$, which is the contraharmonic mean of a and b.

III. CONCLUSION

In this work we illustrate lines which are paralel to one side of triangle related to basic means using the homogenous barycentric coordinates. It is an interesting interpretion of means geometrically. By using such paralel lines it is possible construct more complicated means.

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Abdilkadir Altıntaş, Emirdağ Anadolu Lisesi-Afyonkarahisar-Turkey
e-mail adress: **archimedes26@yandex.com**
Leo Giugiuc, National College, Traian- Drobeta Turnu Severin, Romania
e-mail adress: **leonardgiugiuc@yahoo.com**
Ziya Akça, *Departmen of Mathematics Computer Sciences, Eskişehir Osmangazi University, 26480, Eskişehir, Turkey*
e-mail adress: **zakca@ogu.edu.tr**