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Oscillation and Convengence Properties of Second Order Nonlinear Neutral Delay Difference Equations

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Abstract. *In this paper, we consider the second order nonlinear neutral delay difference equations of the form*

$$\Delta\left[r(n)\left(\Delta\left(x(n)-p(n)x(n-\tau)\right)\right)^{\alpha}\right]+q(n)f\left(x(n-\sigma)\right)=0;\quad n\ge n_0 \qquad (*)$$

We establish sufficient conditions which ensures that every solution of $(*)$ is either oscillatory or tends to zero as $n\to\infty$. We also gives examples to illustrate our results.

Mathematics Subject Classification 2010 **:** *39A10, 39A12*
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I. INTRODUCTION

In this paper, we study the oscillation and asymptotic behaviors of the second order nonlinear neutral delay difference equation

$$\Delta\left[r(n)\left(\Delta\left(x(n)-p(n)x(n-\tau)\right)\right)^{\alpha}\right]+q(n)f\left(x(n-\sigma)\right)=0;\quad n\ge n_0 \qquad (1.1)$$

where Δ is the forward difference operator defined by $\Delta x(n)=x(n+1)-x(n)$, $\{p(n)\}$ is a sequence of nonnegative real numbers, $\{q(n)\}$ is a sequence of nonnegative real numbers and $q(n)$ is not identically zero for large values of n, $\{r(n)\}$ is a sequence of positive real numbers τ and σ are nonnegative integers, $\alpha>0$ is the ratio of two odd integers and $f:R\to R$ is a continuous real valued function.

Throughout the paper the following conditions are assumed to be hold

(i) There exist a constant p_0 such that $0\le p(n)\le p_0<1$;

(ii) u $f(u)>0$ for all $u\ne 0$ and there exists a positive constant k such that $\frac{f(u)}{u^{\alpha}}\ge k$, for all $u\ne 0$.

In addition to the above, we assume that

$$\sum_{n=n_0}^{\infty}\frac{1}{\left(r(n)\right)^{1/\alpha}}=\infty \qquad (1.2)$$

If $\{x(n)\}$ is an eventually positive solution of (1.1), then its associated sequence $\{z(n)\}$ is defined by

$$z(n)=x(n)-p(n)x(n-\tau) \qquad (1.3)$$

By a solution (1.1), we mean a real sequence $\{x(n)\}$ which is defined for $n^*\ge min\{n_0,n_0+\tau,n_0+\sigma\}$ and satisfies (1.1) for $n\ge n^*$. We consider only such solution which are nontrivial for all large n. A solution $\{x(n)\}$ of (1.1) is said to be non oscillatory if the terms x(n) of the sequence are eventually positive or eventually negative. Otherwise it is called oscillatory.

Recently, there has been a lot of interest in the study of oscillatory and asymptotic behaviors of second order difference equations see, for example [1,4-10] and the references cited therein. For the general background of difference equations one can refer to [2,3]

Sternal et al. [8] established sufficient conditions under which every bounded solution of (1.1) is either oscillatory or tends to zero as $n\to\infty$ for the cases $\alpha=1$.

Rath et al. [6] established sufficient conditions which ensures that every solution of (1.1) is oscillatory or tends to zero as $n\to\infty$ for the case $\alpha=1$.

Thandapani et al. [11] consider the second order neutral difference equation
$$\Delta\left(a(n)\Delta\left(y(n) - py^{\alpha}(n-k)\right) + q(n)f(y(n+1) - l)\right) = 0, \quad n \ge n_0 \qquad (1.4)$$

and established sufficient conditions under which every bounded solution of (1.4) is oscillatory and determined sufficient conditions for the existence of positive solution which tends to zero as $n \to \infty$.

Li et al. [5] consider the following second order nonlinear neutral delay differential equation
$$\left(r(t)\left(z^{|}(t)\right)^{\alpha}\right)^{|} + q(t)f\left(x(\sigma(t))\right) = 0, \;\; t \ge t_0 > 0, \qquad (1.5)$$

and established sufficient conditions which ensures that every solution of (1.5) is either oscillatory or tends to zero.

In this paper, our aim is to determine sufficient conditions under which every solution of (1.1) is either oscillatory or tends to zero. Our established results are discrete analogues of some well-known result due to [5].

II. LEMMAS

In this section, we give two lemmas that will be useful for establishing our results.
Lemma 2.1. [4] *If X* and *Y* are positive real numbers and $\lambda > 0$, then

$X^{\lambda} - Y^{\lambda} \ge \lambda Y^{\lambda-1}(X-Y); if\ \lambda \ge 1$

or

$X^{\lambda} - Y^{\lambda} \ge \lambda X^{\lambda-1}(X-Y); \quad if\ 0 < \lambda \le 1.$

There is obviously equality when $\lambda = 1\ or X = Y$.

Lemma 2.2. If $\{x(n)\}$ is an eventually positive solution of the equation (1.1) then z(n) satisfies the following two possible cases:

$(c_1)\ z(n) > 0, \quad \Delta z(n) > 0, \;\; \Delta\left(r(n)\left(\Delta z(n)\right)^{\alpha}\right) \le 0,$

$(c_2)\ z(n) < 0, \quad \Delta z(n) > 0, \;\; \Delta\left(r(n)\left(\Delta z(n)\right)^{\alpha}\right) \le 0,$

for $n \ge n_1$ where $n_1 \ge n_0$ is sufficiently large.

Proof: Since $\{x(n)\}$ is an eventually positive solution of (1.1) there exist an integer $n_1 \ge n_0$ such that
$$x(n) > 0, \quad x(n-\tau) > 0\ and\ x(n-\sigma) > 0 \ \text{ for all } n \ge n_1. \qquad (2.1)$$
It follows from (1.1) and (1.3) that
$$\Delta\left(r(n)\Delta\left(z(n)\right)^{\alpha}\right) \le -k\, q(n)\, x^{\alpha}(n-\sigma) \le 0 \qquad (2.2)$$

This shows that $\left\{r(n)\left(\Delta z(n)\right)^{\alpha}\right\}$ is nonincreasing and of one sign. That is, these exists an integer $n_2 \ge n_1$ such that $\Delta z(n) > 0\ or\ \Delta z(n) < 0$ for all $n \ge n_2$.

If $\Delta z(n) > 0$ for $n \ge n_2$, then we have (c_1) or (c_2). We prove that $\Delta z(n) < 0$ is not possible. If $\Delta z(n) < 0$ for $n \ge n_2$, then $r(n)\left(\Delta z(n)\right)^{\alpha} \le -c < 0$ for $n \ge n_2$,

where $c = -r(n_2)(\Delta z(n)_2)^{\alpha} > 0$. Then, we conclude that
$$z(n) \le z(n_2) - c^{\frac{1}{\alpha}} \sum_{s=n_2}^{n-1} \frac{1}{\left(r(s)\right)^{\frac{1}{\alpha}}}.$$

By virtue of the condition (1.2), $z(n) \to -\infty\ as\ n \to \infty$. We consider now the following two cases separately.

Case 1:
If $\{x(n)\}$ is unbounded, then there exists a sequence $\{n_k\}$ such that $\lim_{k\to\infty} n_k = \infty$ and $\lim_{k\to\infty} x(n_k) = \infty$, where
$x(n_k) = max\{x(s): n_0 \le s \le n_k\}.$

Now from (1.3) we have

$z(n_k) = x(n_k) - p(n_k)x(n_k - \tau) \geq x(n_k)(1 - p(n_k)) > 0,$

which contradicts the fact that $\lim_{n\to\infty} z(n) = -\infty$

Case 2:
If $\{x(n)\}$ is bounded, then $\{z(n)\}$ is also bounded, which contradicts $\lim_{n\to\infty} z(n) = -\infty$. Hence $z(n)$ satisfies one of the cases (c_1) and (c_2). This completes the proof.

Lemma 2.3. If $\{x(n)\}$ is an eventually positive solution of (1.1) and $z(n)$ satisfies the case (c_2) then $lim_{n\to\infty} x(n) = 0.$

Proof: by $z(n) < 0$ and $\Delta z(n) > 0$, we have

$\lim_{n\to\infty} z(n) = l \leq 0,$

where l is a finite constant. That is, $\{z(n)\}$ is bounded. As in the proof of Case1 in Lemma 2.2, $\{x(n)\}$ is also bounded. Using the fact that $\{x(n)\}$ is bounded, we obtain

$lim_{n\to\infty} sup\ \ x(n) = a,\ \ 0 \leq a < \infty.$

We claim that *a=0*. *If* $a > 0$, then there exists a sequence $\{n_k\}$ of integers such that $n_k \to \infty$ and $x(n_k) \to a$ *as* $n \to \infty$.

Let $\varepsilon = \frac{a(1-p_0)}{2p_0}$. Then for all large k, $x(n_k - \tau) < a + \varepsilon$, and so

$$0 \geq \lim_{k\to\infty} z(n_k) \geq \lim_{k\to\infty} x(n_k) - p_0(a + \varepsilon) = \frac{a(1 - p_0)}{2} > 0,$$

which is a contradiction. Thus, *a=0 and* $\lim_{n\to\infty} x(n) = 0$. This completes the proof.

III. OSCILLATION AND CONVERGENCE OF SOLUTIONS TO (1.1)

In what follows we are the following notations:

$$\rho(n) = (r(n))^{\frac{1}{\alpha}} \sum_{s=n_1}^{n-1} \frac{1}{(r(s))^{\frac{1}{\alpha}}},$$

where $n_1 \geq n_0$ is sufficiently large and

$(u(n))_+ = \max\{0, u(n)\}$, where $\{u(n)\}$ is a sequence of real numbers.

Theorem 3.1. Suppose that there exists a sequence $\{\eta(n)\}_{n=n_0}^{\infty}$ of positive real numbers such that for all sufficiently large $n_1 \geq n_0$

$$\sum_{n=n_0}^{\infty} \left[k\ \eta(n)q(n) - \frac{r(n-\sigma)}{\rho^{\alpha}(n-\sigma)}(\Delta\eta(n))_+\right] = \infty. \qquad (3.1)$$

Then every solution of (1.1) is either oscillatory or tends to zero as $n \to \infty$.

Proof : Assume the contrary. Without loss of generality, we may suppose that $\{x(n)\}$ is an eventually positive solution of (1.1). Then there exists an integer $n_1 \geq n_0$ such that $x(n) > 0,\ x(n-\tau) > 0$ and $x(n-\sigma) > 0$ for all $n \geq n_1$. Then from (1.1) and (1.3), we have (2.2). From Lemma 2.2, $z(n)$ satisfies one of the cases (c_1) *or* (c_2). We consider each of two cases separately.

Suppose first that (c_1) holds. By the definition of *z(n)*.

$x(n) = z(n) + p(n)x(n-\tau) \geq z(n)$ (3.2)

By the nonincreasing nature *of* $\left[r(n)\left(\Delta(z(n))\right)^{\alpha}\right]$ we have

$r(n)\left(\Delta z(n)\right)^{\alpha} \leq r(n-\sigma)\left(\Delta z(n-\sigma)\right)^{\alpha}.$ (3.3)

Also by (2.2), we obtain

$$z(n) = z(n_1) + \sum_{s=n}^{n-1} \frac{\left(r(s)\left(\Delta z(s)\right)^{\alpha}\right)^{1/2}}{\left(r(s)\right)^{\frac{1}{\alpha}}} \geq \left(r(n)\right)^{1/\alpha} \Delta z(n) \sum_{s=n_1}^{n-1} \frac{1}{\left(r(s)\right)^{1/\alpha}} = \rho(n)\Delta z(n). \quad (3.4)$$

Let us define a sequence *{ω(n)}* by

$$\omega(n) = \eta(n)\frac{r(n)\left(\Delta z(n)\right)^{\alpha}}{z^{\alpha}(n-\sigma)}, \qquad n \geq n_1. \quad (3.5)$$

Then $\omega(n) > 0$ for $n \geq n_1$ and

$$\Delta\omega(n) = \eta(n)\frac{\Delta\left(r(n)\left(\Delta z(n)\right)^{\alpha}\right)}{z^{\alpha}(n-\sigma)} - \eta(n)\frac{r(n+1)\left(\Delta z(n+1)\right)^{\alpha}}{z^{\alpha}(n+1-\sigma)z^{\alpha}(n-\sigma)}\Delta z^{\alpha}(n-\sigma)$$
$$+\frac{r(n+1)\left(\Delta z(n+1)\right)^{\alpha}}{z^{\alpha}(n+1-\sigma)}\Delta\eta(n). \quad (3.6)$$

Using (2.2), (3.2) and (3.4) in (3.6), we have

$$\Delta\omega(n) \leq -k\eta(n)\rho(n) + \frac{r(n-\sigma)}{\rho^{\alpha}(n-\sigma)}\left(\Delta\eta(n)\right)_{+}. \quad (3.7)$$

Summing the above inequality (3.7) from $n_2 (n_2 > n_1)$ *to n-1*,

we obtain

$$\sum_{s=n_2}^{n-1}\left[k\eta(s)\rho(s) - \frac{r(s-\sigma)}{\rho^{\alpha}(s-\sigma)}\left(\Delta\eta(s)\right)_{+}\right] \leq \omega(n_2), \quad (3.8)$$

which contradicts (3.1).

If *z(n)* satisfies (c_2), then $\lim_{n\to\infty} x(n) = 0$ due to Lemma 2.3. The proof is complete.

Let $\eta(n) = 1$. We can obtain the following criterion for (1.1) using Theorem 3.1.

Corollary 3.2: If

$$\sum_{n=n_0}^{\infty} q(n) = \infty,$$

then every solution of (1.1) is either oscillatory or tends to zero as $n \to \infty$.

Theorem 3.2. Assume that $\alpha \geq 1$. Suppose that there exists a sequence $\{\eta(n)\}_{n=n_0}^{\infty}$ of positive real numbers such that for sufficiently large $n_1 \geq n_0$,

$$\sum_{n=n_0}^{\infty}\left[k\eta(n)q(n) - \frac{\left(\Delta\eta(n)\right)_{+} r(n-\sigma)}{4\alpha\eta(n)\rho^{\alpha-1}(n-\sigma)}\right] = \infty \quad (3.9)$$

Then every solution of (1.1) is either oscillatory or tends to zero as $n \to \infty$.

Proof. Assume the contrary. Without loss of generality, we may suppose that $\{x(n)\}$ is an eventually positive solution of (1.1). Then there exists an integer $n_1 \geq n_0$ such that $x(n) > 0, x(n-\tau) > 0$ *and* $x(n-\sigma) > 0$ for all $n \geq n_1$.

By virtue of Lemma 2.2. *z(n)* satisfies one of (c_1) and (c_2). We discuss each of the two cases separately.

Assume first that *{z(n)}* satisfies *(c_1)* we obtain (3.3) and (3.4). Define the sequence $\{\omega(n)\}$ *by* (3.5).Then $\omega(n) > 0$ for $n \geq n_1$ and we obtain the equation (3.6)

By Lemma 2.1, we have

$$\Delta z^{\alpha}(n-\sigma) = z^{\alpha}(n+1-\sigma) - z^{\alpha}(n-\sigma) \geq \alpha z^{\alpha-1}(n-\sigma)\Delta z(n-\sigma). \qquad (3.10)$$

Using *(3.10)* in *(3.6)*, we obtain

$$\Delta\omega(n) \leq \eta(n)\frac{\Delta\left(r(n)\left(\Delta z(n)\right)^{\alpha}\right)}{z^{\alpha}(n-\sigma)} - \alpha\frac{\eta(n)}{\eta(n+1)}\left(\frac{\Delta z(n-\sigma)}{z(n-\sigma)}\right)\omega(n+1)$$
$$+\frac{\left(\Delta\eta(n)\right)_{+}}{\eta(n+1)}\omega(n+1). \qquad (3.11)$$

Using (2.2) and (3.2) in (3.11), we have

$$\Delta\omega(n) \leq -k\eta(n)q(n) - \alpha\frac{\eta(n)}{\eta(n+1)}\left(\frac{\Delta z(n-\sigma)}{z(n-\sigma)}\right)\omega(n+1) + \frac{\left(\Delta\eta(n)\right)_{+}}{\eta(n+1)}\omega(n+1). \qquad (3.12)$$

On the other hand, by the decreasing nature of $\{r(n)\left(\Delta z(n)\right)^{\alpha}\}$ and *(3.4)*, we have

$$\frac{\Delta z(n-\sigma)}{z(n-\sigma)} = \frac{1}{r(n-\sigma)}\frac{r(n-\sigma)\left(\Delta z(n-\sigma)\right)^{\alpha}}{z^{\alpha}(n-\sigma)}\left(\frac{z(n-\sigma)}{\Delta z(n-\sigma)}\right)^{\alpha-1}$$

$$\geq \frac{1}{\eta(n+1)r(n-\sigma)}\left[\frac{\eta(n+1)r(n+1-\sigma)\left(\Delta z(n+1-\sigma)\right)^{\alpha}}{z^{\alpha}(n+1-\sigma)}\right]\left(\frac{z(n-\sigma)}{\Delta z(n-\sigma)}\right)^{\alpha-1}$$

$$= \frac{\omega(n+1)}{\eta(n+1)r(n-\sigma)}\left[\frac{z(n-\sigma)}{\Delta z(n-\sigma)}\right]^{\alpha-1}$$

$$= \frac{\omega(n+1)}{\eta(n+1)r(n-\sigma)}\rho^{\alpha-1}(n-\sigma). \qquad (3.13)$$

Using (3.13) and (3.12), we obtain

$$\Delta\omega(n) \leq -k\eta(n)q(n) + \frac{\left(\Delta\eta(n)\right)_{+}}{\eta(n+1)}\omega(n+1) - \frac{\alpha\eta(n)\rho^{\alpha-1}(n-\sigma)}{\eta^{2}(n+1)r(n-\sigma)}\omega^{2}(n+1)$$

$$\leq -k\eta(n)q(n) + \frac{r(n-\sigma)\left(\left(\Delta\eta(n)\right)_{+}\right)^{2}}{4\alpha\eta(n)\rho^{\alpha-1}(n-\sigma)}. \qquad (3.14)$$

Summing the above inequality from n_2 $(n_2 > n_1)$ to *n-1*, we obtain

$$\sum_{s=n_2}^{n-1}\left[k\eta(s)q(s) - \frac{r(s-\sigma)\left(\left(\Delta\eta(s)\right)_{+}\right)^{2}}{4\alpha\eta(s)\rho^{\alpha-1}(s-\sigma)}\right] \leq \omega(n_2), \qquad (3.15)$$

which contradicts (3.9).

If z(n) satisfies *(c_2)*, then by Lemma 2.3, $\lim_{n\to\infty} x(n) = 0$ and this completes the proof. ∎

IV. EXAMPLES

Example 4.1.

Consider the following second order neutral delay difference equation

$$\Delta^{2}\left(x(n) - \frac{1}{n+1}x(n-1)\right) + \left[\frac{n+4}{n+3} + \frac{2(n+3)}{n+2} + \frac{n+2}{n+1}\right]x(n-3) = 0; n = 0,1,2,\ldots \quad (4.1)$$

where $r(n) = 1, \alpha = 1, \tau = 1, \sigma = 3,$ $p(n) = \frac{1}{n+1}, q(n) = \frac{n+4}{n+3} + \frac{2(n+3)}{n+2} + \frac{n+2}{n+1}$ and k=1.

It follows from Corollary 3.2 that every solution of (4.1) is either oscillatory or tends to zero as $n \to \infty$. For instance, $x(n) = (-1)^n$ is an oscillatory solution of the equation (4.1)

Example 4.2.
Consider the following second order neutral delay difference equation

$$\Delta\left(n^3\left(\Delta\left(x(n) - \frac{1}{2}x(n-1)\right)\right)^3\right) + \left(1 + \frac{(n-3)^3}{3n}\right)x^3(n-2) = 0; \quad n = 1,2,\ldots. \qquad (4.2)$$

we have $r(n) = n^3, p(n) = \frac{1}{2}, q(n) = 1 + \frac{(n-3)^3}{3n}, \tau = 1, \sigma = 2, \alpha = 3, k = 1.$

Choose $\eta(n) = n.$ Then $\left(\Delta\eta(n)\right)_+ = 1.$ We can easily show that $\rho(n-\sigma) \geq \frac{n}{n-1}$

Then

$$\sum_{n=1}^{\infty}\left[x\eta(n)q(n) - \frac{\left(\Delta\eta(n)\right)_+ r(n-\sigma)}{\rho^\alpha(n-\sigma)}\right] \geq \sum_{n=1}^{\infty}\left[3n\left(1 + \frac{(n-3)^3}{3n}\right) - \frac{(n-2)^3}{\left(\frac{n-2}{n-3}\right)^2}\right]$$

$$= \sum_{n=1}^{\infty} 3n$$

$$= \infty.$$

It follows from Theorem 3.1, every solution of (1.1) is either oscillatory or tend to zero

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