Edge Vertex Prime Labeling for Wheel, Fan and Friendship Graph

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ABSTRACT: In this paper, we investigate edge vertex prime labeling of some graphs. We proved that wheel graph, fan graph and friendship graphs are edge vertex prime.

KEYWORDS: Edge-vertex prime labeling, fan graph, friendship graph, graph labeling, prime labeling, wheel graph.

I. INTRODUCTION

In this paper, all graphs \( G = (V(G), E(G)) \) are simple, undirected and finite. The symbol \( V(G) \) and \( E(G) \) will denote the vertex set and edge set of the graph \( G \). \(|V(G)|\) and \(|E(G)|\) symbols will denote number of vertices and number of edges respectively. For various graph theoretic notation and terminology, I follow J. Gross and J. Yellen [1] and for Number theory D Burton [2]. In this paper, I have proved that wheel graph, fan graph and friendship graph are edge vertex prime.

Definition 1.1:
Graph labeling is an assignment of integers either to the vertices or edges or both subject to certain conditions. A dynamic survey of graph labeling is regularly updated by J Gallian [3].

Definition 1.2:
A bijection function \( f: V(G) \to \{1, 2, ..., n\} \) is called prime labeling if for each edge \( e = uv, \gcd(f(u), f(v)) = 1 \).

A graph which admits prime labeling is called a prime graph.
The notion of a prime labeling was introduced by R Entringer and was discussed in a paper by Tout. A(1982 P 365 – 368)[4].

Definition 1.3:
A bijection function \( f: V(G) \cup E(G) \to \{1, 2, ..., |V(G) \cup E(G)|\} \) is called edgevertexprime labeling if for any edge \( e = uv, f(u), f(v) \) and \( f(uv) \) are pairwise relatively prime.

A graph which admits prime labeling is called an edge vertex prime graph.
Edge vertex prime labeling was introduced by R. Jagadesh and J. BaskarBabujee[5], they proved the existence of the edge vertex prime labeling for paths, cycles and star-\( K_{1,n} \).

Definition 1.4:
Wheel graph \( W_n \) is obtained by joining all vertices of a cycle \( C_n \) to a further vertex, called center.
\[ W_n = K_1 + C_n, |V(G)| = n + 1 \text{ and } |E(G)| = 2n \]

Definition 1.5:
Fan graph \( f_n, n \geq 2 \) obtained by joining all vertices of a path \( P_n \) to a further vertex, called center.
\[ f_n = K_1 + P_n, |V(G)| = n + 1 \text{ and } |E(G)| = 2n - 1 \]

Definition 1.6:
Friendship graph \( F_n \), is a graph which consists of \( n \) – triangles with a common vertex.
\[ |V(G)| = 2n + 1 \text{ and } |E(G)| = 3n \]

II. MAIN RESULTS

Theorem 2.1: Wheel graph is an edge vertex prime.
Proof: Let \( G = W_n \) be the graph.
$V(G) = \{v, v_1, v_2, \ldots, v_n\}$ and
$E(G) = \{v_i/1 \leq i \leq n\} \cup \{v_i v_{i+1}/1 \leq i \leq n\}$
and a bijective function $f: V(G) \cup E(G) \to \{1, 2, \ldots, 3n + 1\}$ is defined as follows:

**Case-1:** When $n$ is even.

$f(v) = 1$

$$f(v_i) = \begin{cases} 
3i & ; \text{i is odd} \\
3i - 1 & ; \text{i is even}
\end{cases}$$

$$f(v v_i) = \begin{cases} 
3i - 1 & ; \text{i is odd} \\
3i & ; \text{i is even}
\end{cases}$$

Now our claims are (1) $f(v), f(v_i)$ and $f(v v_i)$ are pairwise relatively prime.

(2) $f(v_i), f(v_i v_{i+1})$ and $f(v_{i+1})$ are pairwise relatively prime.

1. $\gcd(f(v), f(v_i)) = \begin{cases} 
\gcd(1, 3i) & ; \text{i is odd} \\
\gcd(1, 3i - 1) & ; \text{i is even}
\end{cases}$

$$= 1$$

$$\gcd(f(v), f(v v_i)) = \begin{cases} 
\gcd(1, 3i - 1) & ; \text{i is odd} \\
\gcd(1, 3i) & ; \text{i is even}
\end{cases}$$

$$= 1$$

and $\gcd(f(v_i), f(v v_i)) = \begin{cases} 
\gcd(3i, 3i - 1) & ; \text{i is odd} \\
\gcd(3i - 1, 3i) & ; \text{i is even}
\end{cases}$

$$= 1$$

2. $\gcd(f(v_i), f(v_{i+1})) = \begin{cases} 
\gcd(3i, 3(i + 1) - 1) & ; \text{i is odd} \\
\gcd(3i - 1, 3(i + 1)) & ; \text{i is even}
\end{cases}$

$$= \begin{cases} 
\gcd(3i, 3i + 2) & ; \text{i is odd} \\
\gcd(3i - 1, 3i + 3) & ; \text{i is even}
\end{cases}$$

$$= 1$$

$$\gcd(f(v_i), f(v_i v_{i+1})) = \begin{cases} 
\gcd(3i, 3i + 1) & ; \text{i is odd} \\
\gcd(3i - 1, 3i + 1) & ; \text{i is even}
\end{cases}$$

and $\gcd(f(v_{i+1}), f(v_i v_{i+1})) = \begin{cases} 
\gcd(3(i + 1) - 1, 3i + 1) & ; \text{i is odd} \\
\gcd(3(i + 1), 3i + 1) & ; \text{i is even}
\end{cases}$

$$= \begin{cases} 
\gcd(3i + 2, 3i + 1) & ; \text{i is odd} \\
\gcd(3i + 3, 3i + 1) & ; \text{i is even}
\end{cases}$$

$$= 1$$

Therefore, for any edge $uv \in E(G)$, the numbers $f(u), f(v)$ and $f(u v)$ are pairwise relatively prime.

**Case-2:** When $n$ is odd.

$f(v) = 1$

$$f(v_i) = \begin{cases} 
3i & ; i = 1, 3, 5, \ldots, n - 2 \\
3i - 1 & ; i = 2, 4, 6, \ldots, n - 1
\end{cases}$$

$$f(v v_i) = \begin{cases} 
3i - 1 & ; \text{i is odd} \\
3i & ; \text{i is even}
\end{cases}$$
\[ f(v_1v_{i+1}) = 3i + 1; \quad i = 1, 2, 3, ..., n - 2 \]
\[ f(v_n) = 3n - 2 \]
\[ f(v_{n-1}v_n) = 3n \]

Now our claims are (1) \( f(v), f(v_1), \) and \( f(vv_1) \) are pairwise relatively prime.
(2) \( f(v_i), f(v_i, v_{i+1}) \) and \( f(v_{i+1}) \) are pairwise relatively prime.

\[(1) \quad \gcd(f(v), f(v_i)) = \begin{cases} \gcd(1, 3i) & ; \; i = 1, 3, 5, ..., n - 2 \\ \gcd(1, 3i - 1) & ; \; i = 2, 4, 6, ..., n - 1 \\ \gcd(1, 3n - 2) & ; \; i = n \end{cases} \]
= 1

\[ \gcd(f(v), f(vv_1)) = \begin{cases} \gcd(1, 3i - 1) & ; \; i \text{ is odd} \\ \gcd(1, 3i) & ; \; i \text{ is even} \end{cases} \]
= 1

\[ \text{And } \gcd(f(v_i), f(vv_i)) = \begin{cases} \gcd(3i, 3i - 1) & ; \; i = 1, 3, 5, ..., n - 2 \\ \gcd(3i - 1, 3i) & ; \; i = 2, 4, 6, ..., n - 1 \\ \gcd(3n - 2, 3n + 1) & ; \; i = n \end{cases} \]
= 1

\[(2) \quad \gcd(f(v_i), f(v_{i+1})) = \begin{cases} \gcd(3i, 3i + 2) & ; \; i = 1, 3, 5, ..., n - 2 \\ \gcd(3i - 1, 3i + 3) & ; \; i = 2, 4, 6, ..., n - 1 \\ \gcd(3n - 2, 3n + 1) & ; \; i = n - 1 \end{cases} \]
= 1

\[ \gcd(f(v_i), f(vv_{i+1})) = \begin{cases} \gcd(3i, 3i + 1) & ; \; i = 1, 3, 5, ..., n - 2 \\ \gcd(3i - 1, 3i + 1) & ; \; i = 2, 4, 6, ..., n - 1 \\ \gcd(3n - 1, 3n + 1) & ; \; i = n - 1 \end{cases} \]
= 1

\[ \text{And } \gcd(f(v_{i+1}), f(vvv_{i+1})) = \begin{cases} \gcd(3(i + 1) - 1, 3i + 1) & ; \; i = 1, 3, 5, ..., n - 2 \\ \gcd(3(i + 1), 3i + 1) & ; \; i = 2, 4, 6, ..., n - 3 \\ \gcd(3n - 2, 3n) & ; \; i = n - 1 \end{cases} \]
= 1

Therefore, for any edge \( uv \in E(G) \), the numbers \( f(u), f(v) \) and \( f(uv) \) are pairwise relatively prime.
Hence \( W_n \) is an edge vertex prime graph.
Illustration 2.1: Edge vertex prime labeling for $W_{28}$.

Illustration 2.2: Edge vertex prime labeling for $W_{27}$.

Theorem 2.2: Fan graph is an edge vertex prime

Proof: Let $G = f_n$, $n \geq 2$ be the graph.

$V(G) = \{v, v_1, v_2, \ldots, v_n\}$ and

$E(G) = \{vv_i/1 \leq i \leq n\} \cup \{v_i v_{i+1}/1 \leq i \leq n - 1\}$

and a bijective function $f: V(G) \cup E(G) \to \{1, 2, \ldots, 3n\}$ is defined as follows:

$f(v) = 1$

$f(v_i) = \begin{cases} 3i &; i \text{ is odd} \\ 3i - 1 &; i \text{ is even} \end{cases}$

$f(vv_i) = \begin{cases} 3i - 1 &; i \text{ is odd} \\ 3i &; i \text{ is even} \end{cases}$
Now our claims are (1) \( f(v), f(v_i) \) and \( f(vv_i) \) are pairwise relatively prime.

(2) \( f(v_i), f(v_i+1) \) and \( f(v_{i+1}) \) are pairwise relatively prime.

(1) \[ \gcd(f(v), f(v_i)) = \begin{cases} \gcd(1, 3i) ; & \text{if } i \text{ is odd} \\ \gcd(1, 3i - 1) ; & \text{if } i \text{ is even} \end{cases} \]

\[ = 1 \]

\[ \gcd(f(v), f(vv_i)) = \begin{cases} \gcd(1, 3i - 1) ; & \text{if } i \text{ is odd} \\ \gcd(1, 3i) ; & \text{if } i \text{ is even} \end{cases} \]

\[ = 1 \]

and \[ \gcd(f(v_i), f(vv_i)) = \begin{cases} \gcd(3i, 3i - 1) ; & \text{if } i \text{ is odd} \\ \gcd(3i - 1, 3i) ; & \text{if } i \text{ is even} \end{cases} \]

\[ = 1 \]

(2) \[ \gcd(f(v_i), f(v_{i+1})) = \begin{cases} \gcd(3i, 3i + 1) ; & \text{if } i \text{ is odd} \\ \gcd(3i - 1, 3i + 1) ; & \text{if } i \text{ is even} \end{cases} \]

\[ = 1 \]

\[ \gcd(f(v_i), f(vv_{i+1})) = \begin{cases} \gcd(3i, 3i + 1) ; & \text{if } i \text{ is odd} \\ \gcd(3i - 1, 3i + 1) ; & \text{if } i \text{ is even} \end{cases} \]

\[ = 1 \]

and \[ \gcd(f(v_{i+1}), f(vv_{i+1})) = \begin{cases} \gcd(3i + 1, 3i + 1) ; & \text{if } i \text{ is odd} \\ \gcd(3i, 3i + 1) ; & \text{if } i \text{ is even} \end{cases} \]

\[ = 1 \]

Therefore, for any edge \( uv \in E(G) \), the numbers \( f(u), f(v) \) and \( f(uv) \) are pairwise relatively prime. Hence fan graph is an edge vertex prime.

**Illustration 2.3:** Edge vertex prime labeling for \( f_{29} \).

**Theorem 2.3:** Friendship graph is an edge vertex prime

**Proof:** Let \( G = F_n \) be the graph.

\[ V(G) = \{v, v_1, v_2, \ldots, v_{2n}\} \]

\[ E(G) = \{vv_i/1 \leq i \leq 2n\} \cup \{v_{2i-1}v_{2i}/1 \leq i \leq n\} \]

and a bijective function \( f: V(G) \cup E(G) \rightarrow \{1, 2, \ldots, 5n + 1\} \) is defined as follows:

\( f(v) = 1 \)
$f(v_{2i-1}) = \begin{cases} 5i - 2 & \text{; } i \text{ is odd} \\ 5i - 3 & \text{; } i \text{ is even} \end{cases}$

$f(v_{2i}) = \begin{cases} 5i & \text{; } i \text{ is odd} \\ 5i + 1 & \text{; } i \text{ is even} \end{cases}$

$f(vv_{2i-1}) = \begin{cases} 5i - 3 & \text{; } i \text{ is odd} \\ 5i - 2 & \text{; } i \text{ is even} \end{cases}$

$f(vv_{2i}) = \begin{cases} 5i + 1 & \text{; } i \text{ is odd} \\ 5i & \text{; } i \text{ is even} \end{cases}$

$f(v_{2i-1}v_{2i}) = 5i - 1 \quad \forall i$

Now our claims are (1) $f(v), f(v_{2i-1})$ and $f(vv_{2i-1})$ are pairwise relatively prime.

(2) $f(v), f(v_{2i})$ and $f(vv_{2i})$ are pairwise relatively prime.

(3) $f(v_{2i-1}), f(v_{2i-1}v_{2i})$ and $f(v_{2i})$ are pairwise relatively prime.

(1) $gcd\left(f(v), f(v_{2i-1})\right) = \begin{cases} gcd(1,5i - 2) & \text{; } i \text{ is odd} \\ gcd(1,5i - 3) & \text{; } i \text{ is even} \end{cases} = 1$

$gcd\left(f(v), f(vv_{2i-1})\right) = \begin{cases} gcd(1,5i - 3) & \text{; } i \text{ is odd} \\ gcd(1,5i - 2) & \text{; } i \text{ is even} \end{cases}$

and $gcd\left(f(v_{2i-1}), f(vv_{2i-1})\right) = \begin{cases} gcd(5i - 2, 5i - 3) & \text{; } i \text{ is odd} \\ gcd(5i - 3, 5i - 2) & \text{; } i \text{ is even} \end{cases} = 1$

(2) $gcd\left(f(v), f(v_{2i})\right) = \begin{cases} gcd(1,5i) & \text{; } i \text{ is odd} \\ gcd(1,5i + 1) & \text{; } i \text{ is even} \end{cases} = 1$

$gcd\left(f(v), f(vv_{2i})\right) = \begin{cases} gcd(1,5i + 1) & \text{; } i \text{ is odd} \\ gcd(1,5i) & \text{; } i \text{ is even} \end{cases} = 1$

and $gcd\left(f(v_{2i}), f(vv_{2i})\right) = \begin{cases} gcd(5i, 5i + 1) & \text{; } i \text{ is odd} \\ gcd(5i + 1,5i) & \text{; } i \text{ is even} \end{cases} = 1$

(3) $gcd\left(f(v_{2i-1}), f(v_{2i})\right) = \begin{cases} gcd(5i - 2, 5i) & \text{; } i \text{ is odd} \\ gcd(5i - 3, 5i + 1) & \text{; } i \text{ is even} \end{cases} = 1$

$gcd\left(f(v_{2i-1}), f(v_{2i-1}v_{2i})\right) = \begin{cases} gcd(5i - 2, 5i - 1) & \text{; } i \text{ is odd} \\ gcd(5i + 1,5i - 1) & \text{; } i \text{ is even} \end{cases} = 1$

and $gcd\left(f(v_{2i}), f(v_{2i-1}v_{2i})\right) = \begin{cases} gcd(5i, 5i - 1) & \text{; } i \text{ is odd} \\ gcd(5i + 1,5i - 1) & \text{; } i \text{ is even} \end{cases} = 1$

Therefore, for any edge $uv \in E(G)$, the numbers $f(u), f(v)$ and $f(uv)$ are pairwise relatively prime.
Hence Friendship graph $F_n$ is an edge vertex prime.

**Illustration 2.4:** Edge vertex prime labeling for $F_{15}$.

![Fig. 2.4 $F_{15}$](image)

**REFERENCES**


