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A Non-Uniform Bound Approximation of Polya via Poisson, Using Stein-Chen Method and Ω –Function and Its Application in Option Pricing

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ABSTRACT : *In this work, the Stein-Chen method and the ω-function associated with Pólya random variable are used to obtain a non-uniform bound approximation of Pólya by Poisson in terms of point metric and Pólya distribution applied in finance. Egege et al[11]found out that the upper bound may not be sufficiently enough for measuring the accuracy of an approximation. Also, it was discovered that the results obtained for non-uniform are better than the results obtained for uniform bound. This is in agreement with Egege et al,[11], that for $n \leq r$, and if the upper bound is small, then a good approximation of Pólya is obtained. And for n>r , it will not give an appropriate approximation. And Pólya combined with financial term is used to generate a model for forecasting the price of a European call option. Which gives the same numerical with Osu ,et al[2].*
***KEYWORDS** -Stein-Chen equation, Pólya distribution, Option Pricing, Non-uniform Bound Approximation.*

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I. INTRODUCTION

The aim of this work is to give a non-uniform bound approximation of Pólya by Poisson in terms of point metric in extension to Samson et al, and associating Pólya distribution with financial terms to generate a model for evaluating the price of an option.

This paper focuses on a particular type of derivative security known as an option. Determining an option value is called option pricing. Cheng-Few et al [2] showed how the Binomial distribution is combined with some basic finance concepts to generate a model for determining the price of stock option to be:

$$C = \frac{1}{R^n}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}p^k(1-p)^{n-k}max[0, u^k d^{n-k}S - K]. \qquad (1)$$

Osu et al[2] develop a model using generalized binomial distribution of the form

$$C_{(0)} = \frac{1}{R^n}\sum_{x=0}^{n}\binom{n}{x}\frac{\hat{A}^x\hat{B}^{n-x}}{(\hat{A}+\hat{B})^{(n)}}Max\left[u^x d^{n-x}S_{(0)} - K, 0\right], \qquad (2)$$

where this can also be expressed as of the form

$$C_{(0)} = \frac{1}{R^n}\sum_{x=0}^{n}\binom{n}{x}\frac{\hat{A}^x\hat{B}^{n-x}}{(\hat{A}+\hat{B})^{(x)}(\hat{A}+\hat{B})^{(n-x)}}C_t(x). \qquad (3)$$

Where $C_T(x) = Max\left[u^x d^{n-x}S_{(0)} - K, 0\right]$,$R$ is the interest rate ,$\frac{\hat{A}}{\hat{A}+\hat{B}}$ and $\frac{\hat{B}}{\hat{A}+\hat{B}}$ are the neutral probability,u and $d$are the rates at which the price move up and down respectively and k is the strike price.

Osu et al[2] used the generalized binomial model to evaluate the price of call and put options.

The proposed model is of the form

$$C_{(0)} = \frac{1}{(1+r)^n}\sum_{x=0}^{n}\binom{n}{x}\frac{\left(\check{r}/_{\check{r}+\breve{b}}\right)^x\left(\breve{b}/_{\check{r}+\breve{b}}\right)^{n-x}}{\left(\check{r}/_{\check{r}+\breve{b}}+\frac{\breve{b}}{\check{r}+\breve{b}}\right)^n}C_T(x) \quad (4)$$

where $C_T(x) = \max\left[u^x d^{n-x}S_{(0)} - K\right]$, $(1+r)$ is the interest rate ,$\frac{\check{r}}{\check{r}+\breve{b}}$ and $\frac{\breve{b}}{\check{r}+\breve{b}}$ is the neutral probabilities

ThePólya distribution in this work was first studied and presented by G. Pólya(1923).It is a discrete distribution that depends on four parameters N, n, r, cwritten in the form $\mathbb{P}_{\mathbb{y}}(N, n, r, c)$ where the integers $n > 0$,N, n, r, c are real numbers ,$0 < \frac{r}{r+b} < 1$,$b = N - r$ and $c > 0$ are parameters

The details of the background of this distribution can been in Feller [5].Let Xbe Pólya random variable, then its probability function is of the form

$$P_\chi(x) = \frac{\binom{\frac{r}{c}+x-1}{x}\binom{\frac{N-r}{c}+n-x-1}{n-x}}{\binom{\frac{N}{c}+n-1}{n}}x = 0,1, \ldots . n. \qquad (5)$$

Then the mean and variance of X are $\mu = \frac{nr}{N}$ and $\sigma^2 = \frac{nr\,(N+cn)(N-r)}{N^2(N+c)}$ respectively.Samson et al[11] considered a special case of $c = 1$, and showed that $\mathbb{P}_{\mathbb{y}}(N, n, r, c)$ converges to Binomial distribution with parameters $n, \frac{r}{r+b}$ which is denoted by$\mathbb{B}\left(n, \frac{r}{r+b}\right)$and its distribution function is of the form

$P_\chi(x) \to \binom{n}{x}\left({r}/{r+b}\right)^x\left({b}/{r+b}\right)^{n-x} \quad \forall x = 0,1, \dots . n. \qquad (6)$

In view of (2), it is clear that $\mathbb{P}_{\mathbb{y}}(N, n, r, c)$ can also be approximated by$\mathbb{B}\left(n, \frac{r}{r+b}\right)$ under appropriate conditions on their parameters. For example K. Teerapabolan[7]used Stein-Chen method and ω-function associated with Pólya random variable to give a bound for total variation distance between the Binomial and Pólya distribution of the form

$d\left(\left(\mathbb{B}(np), \mathbb{P}_{\mathbb{y}}(N\ , n\ , r\ , c)\right)\right) \le \frac{(1-P^{n+1}-q^{n+1})c(n\_1)n}{(n+1)(N+1)}. \qquad (7)$

Where $N\ , n\ , rc$ are non-negative valued integers and $\mathbb{B}(np) = \binom{n}{x}p^n q^{n-1}$,

$$\mathbb{P}_{\mathbb{y}}(N\ , n\ , r\ , c) = \frac{\binom{\frac{r}{c}+x-1}{x}\binom{\frac{N-r}{c}+n-x-1}{n-x}}{\binom{\frac{N}{c}+n-1}{n}}$$

is the Pólya distribution.

It is a well-known fact that Poisson and Binomial distributions can be approximated by each other under appropriate conditions on their parameters.

If Binomial distribution can be approximated by Pólya (4) under appropriate conditions on their parameters, thus if the conditions on the parameters of Pólya and Poisson are satisfied, then Pólya can be approximated by Poisson distribution. Egege et al[11] used Poisson to approximate Pólya distribution, in terms of point metric of the form

$| P_{\mathbb{y}}(x_0) - \wp_\lambda(x_0) | \quad \le \left(1 - e^{-\lambda}\right)\frac{r(r+b+c)-b(n-1)c}{(r+b)(r+b+c)}, \qquad (8)$

where$x\epsilon\mathbb{N}$, $\wp_{\boldsymbol{\lambda}}(\boldsymbol{x_0}) = \frac{\boldsymbol{e^{-\lambda}\lambda^{x_0}}}{\boldsymbol{x_0}!}$, $\boldsymbol{\lambda} = \frac{\boldsymbol{nr}}{\boldsymbol{r+b}}$and $(n-1)c \le r$.It can observed that the bound in (6) and the point metric does not depend on x_0, Where$x_0 \in \{0,1 \dots n\}$,the above bound is with respect to x_0 where $x_0 \in \{1,2 \dots\} / \{0\}$

The question here is what will happen if (8) depends on x_0 or with respect to x_0?

The question is what the work seeks to ascertain. The bound (8) may not be sufficiently enough for measuring the accuracy of the approximations.

II. METHODS

The tools for giving the results are, Pólya distribution, wealth equation, Stein-Chen equation, and ω −function.The Pólya distribution is given by G. Pólya 1923 in connection with the so-called Pólya scheme.

The Pólya distribution is a discrete distribution that depends on four parameters N, n, r, c .Denoted by $\mathbb{P}_{\mathbb{y}}(N, n, r, c)$where N, $n, r$are non-negative values and $c = 1$, that satisfies $(n-1)c \le N$.

Let X be the Pólya random variable, G. Pólya gave its probability function of the form

$P_\chi(x) = \frac{\binom{\frac{r}{c}+x-1}{x}\binom{\frac{N-r}{c}+n-x-1}{n-x}}{\binom{\frac{N}{c}+n-1}{n}}, \quad x = 0,1, \dots . n. \qquad (9)$

The mean and variance of X are given respectively by $\mu = \frac{nr}{N}$ and $\sigma^2 = \frac{nr\,(N+cn)(N-r)}{N^2(N+c)}$. Considering a special case of $c = 1$, Maths Planet 2012 equation (1) can be expressed as of the form

$P_\chi(x) = \binom{n}{k}\frac{(r,c)_{x-1}(N-r,c)_{n-x-1}}{(N,c)_{n-1}}, x = 0,1 \dots n, \qquad (10)$

where$(r, c)_{x-1} = r(r+c) \dots \dots . r + (x-1)c$, $(N-r, c)_{n-x-1}$=$N - r(N - r\ + c) \dots \dots N - r + (n - x - 1)c$ and $(N, c)_{n-1} = N(N+c) \dots \dots . . N + (n-1)c$.

Let $N = r + b$and $N - r = b$ we obtain

$$P_X(x) = \binom{n}{x}\frac{[r(r+c) \dots . r + (x-1)c][b(b+c) \dots b + (n-x-1)c]}{[(r+b)(r+b+c) \dots \dots r + b + (n-1)c]}$$

$$= \binom{n}{x}\frac{[r \dots \dots . r + (x-1)c][b \dots . b + (n-x-1)c]}{[r + b \dots . r + b + (n-1)c]}$$

$= \binom{n}{x}\frac{[{r}/{c} \dots\dots {r}/{c}+\xi][{b}/{c} \dots\dots {b}/{c}+(n-x-1)]}{[{r+b}/{c} \dots\dots {r+b}/{c}+(n-1)]}. \qquad (11)$

where

$\xi = \begin{cases} 0 \quad if x = 0 \\ (x-1) if x = 1,2,\ldots..n \end{cases}$ and $\delta = \begin{cases} 0 \, if x = 0,1,\ldots n \\ (n-x-1) if x = 0 \end{cases}$.

Equation (3) can further be simplified to give

$$P_{\chi}(x) = \binom{n}{x} \frac{r+b\left[{}^{r}/_{r+b} \cdots\cdots \left({}^{r}/_{r+b} + {}^{\xi}/_{r+b/_{c}}\right)\right] r+b\left[{}^{b}/_{r+b} \cdots\cdots \left({}^{b}/_{r+b} + {}^{\delta}/_{r+b/_{c}}\right)\right]}{r+b\left[1 \ldots\ldots 1 + {}^{n-1}/_{r+b/_{c}}\right]}$$

$$= \binom{n}{x} \frac{(r+b)^{x}\left[{}^{r}/_{r+b} \cdots\cdots \left({}^{r}/_{r+b} + {}^{\xi}/_{r+b/_{c}}\right)\right](r+b)^{n}\left[{}^{b}/_{r+b} \cdots\cdots \left({}^{\delta}/_{r+b/_{c}}\right)\right]}{(r+b)^{n}\left(1 \ldots\ldots 1 + {}^{n-1}/_{r+b/_{c}}\right)} =$$

$$\binom{n}{x} \frac{\left[(r+b)^{x}(r+b)^{n-x}\left[{}^{r}/_{r+b} \cdots\cdots \left({}^{r}/_{r+b} + {}^{\xi}/_{r+b/_{c}}\right)\right]{}^{b}/_{r+b} \cdots\cdots \left({}^{b}/_{r+b} + {}^{\delta}/_{r+b/_{c}}\right)\right]}{(r+b)^{n}\left[1 \ldots\ldots 1 + {}^{n-1}/_{r+b/_{c}}\right]}. \quad (12)$$

Note that $\frac{(r+b)^{x}(r+b)^{n-x}}{(r+b)^{n}} = \frac{(r+b)^{n}}{(r+b)^{n}} = 1.$

Thus $P_{\chi}(x) = \binom{n}{x} \frac{\left[\left[{}^{r}/_{r+b} \cdots\cdots \left({}^{r}/_{r+b} + {}^{\xi}/_{r+b/_{c}}\right)\right]{}^{b}/_{r+b} \cdots\cdots \left({}^{b}/_{r+b} + {}^{\delta}/_{r+b/_{c}}\right)\right]}{\left[1 \ldots\ldots 1 + {}^{n-1}/_{r+b/_{c}}\right]}$

If $r, r+b \to \infty$, while $\frac{r}{r+b}$ remains constant, implies $\frac{b}{r+b}$ is constant then

$P_{\chi}(x) = \binom{n}{x} \frac{\left({}^{r}/_{r+b}\right)^{x}\left({}^{b}/_{r+b}\right)^{n-x}}{\left({}^{r}/_{r+b} + \frac{b}{r+b}\right)^{n}}$ (13)

It can be seen below that the parameters in satisfy the following conditions

i. $\frac{r}{r+b} + \frac{b}{r+b} = 1$
ii. $1 - \frac{r}{r+b} = \frac{b}{r+b}$
iii. $0 \leq \frac{r}{r+b} \leq 1$Type equation here.

In this work $\frac{r}{r+b}$ and $\frac{b}{r+b}$ are regarded as the neutral probability denoted by $\frac{\check{r}}{\check{r}+\check{b}}$ and $\frac{\check{b}}{\check{r}+\check{b}}$.

Thus Pólya distribution applied in finance is of the form

$$P_{\chi}(x) = \binom{n}{x} \frac{[\check{r}(\check{r}+c) \ldots . \check{r} + (x-1)c][\check{b}(\check{b}+c) \ldots \check{b} + (n-x-1)c]}{[(\check{r}+\check{b})(\check{r}+\check{b}+c) \ldots\ldots \check{r}+\check{b}+(n-1)c]}$$

$$= \binom{n}{x} \frac{\left[\left[{}^{\check{r}}/_{\check{r}+\check{b}} \cdots\cdots \left({}^{\check{r}}/_{\check{r}+\check{b}} + {}^{\xi}/_{\check{r}+\check{b}/_{c}}\right)\right]{}^{\check{b}}/_{\check{r}+\check{b}} \cdots\cdots \left({}^{\check{b}}/_{\check{r}+\check{b}} + {}^{\delta}/_{\check{r}+\tilde{b}/_{c}}\right)\right]}{\left[1 \ldots\ldots 1 + {}^{n-1}/_{\check{r}+\check{b}/_{c}}\right]}$$

$$= \binom{n}{x} \frac{\left({}^{\check{r}}/_{\check{r}+\check{b}}\right)^{x}\left({}^{\check{b}}/_{\check{r}+\check{b}}\right)^{n-x}}{\left({}^{\check{r}}/_{\check{r}+\check{b}} + \frac{\check{b}}{\check{r}+\check{b}}\right)^{n}} x = 0,1,2..n. \quad (14)$$

The Stein-Chen method is a general method in probability theory that isused to obtain bounds on distance between two probabilitydistributionswith respect to a probability metric.

Teerepabolan[6] gave a Stein-Chen equation for Poisson distribution with $\lambda > 0$ and for h of the form.

$h_{\Omega}(x) - \wp_{\lambda}(h) = \lambda f(x+1) - x f(x)$ (15)

Where $\wp_{\lambda}(h) = \sum_{i=0}^{\infty} h(l) \frac{e^{-\lambda}\lambda^{l}}{l!}$and f, h are bound real-valued function defined in $\mathbb{N} \cup \{0\}$.

For$\Omega \subseteq \mathbb{N} \cup \{0\}$, defining an indicator function $h_{x_0}(x) = \mathbb{N} \cup (0) \to \mathbb{R}$ such that

$$h_{x_0}(x)\begin{cases}1 & if x_0 \le x\\ 0 & if x > x_0\end{cases}.$$

For $\Omega = \{x_0\}$ where$x_0 \in \mathbb{N} \cup \{0\}$,Barbour et al gave a solution of (5) of the form by letting $f = f_{x_0}$

$$f_{x_0}(x) = \begin{cases} -(x-1)!\,\lambda^{-x}\left[\wp_\lambda(h_\Omega)\wp_\lambda\left(h_{C_{x-1}}\right)\right] if x \le x_0\\ (x-1)!\,\lambda^{-x}e^{\lambda}\left[\wp_\lambda(h_\Omega)\wp_\lambda\left(1-h_{C_{x-1}}\right)\right] if x > x_0,\\ 0 \qquad if x = 0\end{cases} \quad (16)$$

where $C_x = (0,1, \dots .n)$.

Let $\Delta f_{x_0}(x) = f_{x_0}(x+1) - f_{x_0}(x)$,then the following lemma gives a non-uniform bound of $|\Delta f_{x_0}|$.

For the model, the following assumptions hold

Assumptions

i. The initial value of the stock is $S_{(0)}$($S_{(0)}$ is the stock price at t=0).

ii. At the end of the period, the price is either going up or down with factors $u$and $d$that is, $uS_{(0)}$ with probability $\frac{\hat{A}}{\hat{A}+\hat{B}}$ or $dS_{(0)}$ with probability $\frac{\hat{B}}{\hat{A}+\hat{B}}$ which satisfies $0 < \frac{\hat{A}}{\hat{A}+\hat{B}} < 1$.

iii. The movement can also be traced from a view point of tossing a die,which result to a head and tail if it result to a head at a time, one we have $S_1(H) = uS_{(0)}$and if it result to a tail at a time, one we have $S_1(T) = dS_{(0)}$.

iv. One dollar invested in the money market at time zero will yield $1+r$ dollar at time one,where r is the interest rate .Conversely, one dollar borrowed from the money market at time zero will result in a debt of $1+r$ at time one.

v. The price either increases, by $u > 1$ or will decrease by$d < 1$.

Lemma 1: If no arbitrage principle holds then

$C_{t+1} = V_{t+1} for \forall t \in \{0,1 \dots .T\}$.

Lemma 2

For a risk neutral probability$\frac{\check{r}}{\check{r}+\breve{b}} = \left(\frac{R-u}{u-d}\right)$, and no arbitrage principle $u > 1+r > d > 0$ exist if the following holds

a. $E_{\frac{\check{r}}{\check{r}+\breve{b}}}\left(S_{(2)}\right) = (1+r)^2 S_{(0)}$ where $(1+r)$ denote the interest rate

b. $\sum_{l=1}^{3} \frac{\check{r}}{\check{r}+\breve{b}_l} = 1 \forall l = 1,2,3$

c. $\frac{\check{r}}{\check{r}+\breve{b}_i} > 0$

Proof

For $S_{(2)}$ implies $t = 2$and $\left(\frac{\check{r}}{\check{r}+\breve{b}} + 1 - \frac{\check{r}}{\check{r}+\breve{b}}\right)^2 = \left(\frac{\check{r}}{\check{r}+\breve{b}}\right)^2 + 2\frac{\check{r}}{\check{r}+\breve{b}}\left(1-\frac{\check{r}}{\check{r}+\breve{b}}\right) + \left(1-\frac{\check{r}}{\check{r}+\breve{b}}\right)^2$

Defining $\frac{\check{r}}{\check{r}+\breve{b}_1} = \left(\frac{\check{r}}{\check{r}+\breve{b}}\right)^2$, $\frac{\check{r}}{\check{r}+\breve{b}_2} = 2\frac{\check{r}}{\check{r}+\breve{b}}\left(1-\frac{\check{r}}{\check{r}+\breve{b}}\right)$ and $\frac{\check{r}}{\check{r}+\breve{b}_3} = \left(1-\frac{\check{r}}{\check{r}+\breve{b}}\right)^2$

Now $E_{\frac{\check{r}}{\check{r}+\breve{b}_i}}\left(S_{(2)}\right) = \left[\left(\frac{\check{r}}{\check{r}+\breve{b}}\right)^2 u^2 S_{(0)} + 2\frac{\check{r}}{\check{r}+\breve{b}}\left(1-\frac{\check{r}}{\check{r}+\breve{b}}\right)udS_{(0)} + \left(1-\frac{\check{r}}{\check{r}+\breve{b}}\right)^2 dS_{(0)}\right]$

$$E_{\frac{\check{r}}{\check{r}+\breve{b}_i}}\left(S_{(2)}\right) = \left[\left(\frac{\check{r}}{\check{r}+\breve{b}}\right)^2 u^2 + 2\frac{\check{r}}{\check{r}+\breve{b}}\left(1-\frac{\check{r}}{\check{r}+\breve{b}}\right)ud + \left(1-\frac{\check{r}}{\check{r}+\breve{b}}\right)^2 d\right]S_{(0)}$$

$$= S_{(0)}\left[\frac{\check{r}}{\check{r}+\breve{b}}u + \left(1-\frac{\check{r}}{\check{r}+\breve{b}}\right)d\right]^2$$

$$= S_{(0)}\left[\frac{R-d}{U-d}u + \left(1-\frac{u-R}{u-d}\right)d\right]^2 = S_{(0)}\left[\frac{Ru-du+du-Rd}{u-d}\right]^2 = S_{(0)}\left[\frac{Ru-Rd}{u-d}\right]^2 = S_{(0)}R^2\left[\frac{u-d}{u-d}\right]^2$$

$$E_{\frac{\check{r}}{\check{r}+\breve{b}_i}}\left(S_{(2)}\right) = S_{(0)}(1+r)^2$$

where$(1+r)$ is the interest rate

Defining $\frac{\check{r}}{\check{r}+\breve{b}_1} = \left(\frac{\check{r}}{\check{r}+\breve{b}}\right)^2$, $\frac{\check{r}}{\check{r}+\breve{b}_2} = 2\frac{\check{r}}{\check{r}+\breve{b}}\left(1-\frac{\check{r}}{\check{r}+\breve{b}}\right)$ and $\frac{\check{r}}{\check{r}+\breve{b}_3} = \left(1-\frac{\check{r}}{\check{r}+\breve{b}}\right)^2$we have

$$\sum_{l=1}^{3}\left(\frac{\check{r}}{\check{r}+\breve{b}}\right)^2 + 2\frac{\check{r}}{\check{r}+\check{r}}\left(1-\frac{\check{r}}{\check{r}+\breve{b}}\right) + \left(1-\frac{\check{r}}{\check{r}+\breve{b}}\right)^2.$$

$$\sum_{i=1}^{3}\frac{\check{r}}{\check{r}+\breve{b}_i} = \left[\left(\frac{\check{r}}{\check{r}+\breve{b}}\right)^2 + 2\frac{\check{r}}{\check{r}+\breve{b}} - 2\left(\frac{\check{r}}{\check{r}+\breve{b}}\right)^2 + \left(1-\frac{\breve{b}}{\check{r}+\breve{b}}\right)^2\right] =$$

$$\left[\left(\frac{\check{r}}{\check{r}+\breve{b}}\right)^2 + 2\frac{\check{r}}{\check{r}+\breve{b}} - 2\left(\frac{\check{r}}{\check{r}+\breve{b}}\right)^2 + 1 - 2\frac{\check{r}}{\check{r}+\breve{b}} + \left(\frac{\check{r}}{\check{r}+\breve{b}}\right)^2\right] = \left[\left(\frac{\check{r}}{\check{r}+\breve{b}}\right)^2 - \left(\frac{\check{r}}{\check{r}+\breve{b}}\right)^2 + 1\right] = 1$$

$$\sum_{i=1}^{3}\frac{\check{r}}{\check{r}+\check{b}_i} = 1$$

Now it can be clearly seen that $\frac{\check{r}}{\check{r}+\check{b}_i} > 0$.

Lemma 3: For $\frac{\check{r}}{\check{r}+\check{b}} = \left(\frac{R-u}{u-d}\right)$, and $\frac{\check{b}}{\check{r}+\check{b}} = \frac{u-1-r}{u-d}$ for any given $u > 1 and d < 1$ factors

$S_0 = \frac{1}{1+r}\left[\frac{\check{r}}{\check{r}+\check{b}}uS_1 + \frac{\check{b}}{\check{r}+\check{b}}dS_1\right]$ (17)

Lemma 2.1

For $x_0 \in \mathbb{N}\cup\{0\}, x \in \mathbb{N}$ with $\Delta f_{x_0}(x) = f_{x_0}(x+1) - f_{x_0}(x)$ then the following holds.

$\left|\Delta f_{x_0}\right| \le \lambda^{-1}\left(1-e^{-\lambda}\right)$ (18)

Proof

For $\forall x \in \mathbb{N}$ and $x_0 \in \mathbb{N}\cup\{0\}$, $|\Delta f_{x_0}(x)| = |f_{x_0}(x+1) - f_{x_0}(x)| \le |f_{x_0}(1)|$

Implies $\left|\Delta f_{x_0}(x)\right| \le |f_{x_0}(1)| \forall x \in \mathbb{N}$

Samson Egege et al [11] showed that $|f_{x_0}(1)| = \lambda^{-1}\left(1-e^{-\lambda}\right)$

$$\Rightarrow \left|\Delta f_{x_0}\right| \le \lambda^{-1}\left(1-e^{-\lambda}\right)$$

Lemma 2.2. Let $x_0 \in \mathbb{N}\cup\{0\}, x \in \mathbb{N}$ then the following holds

$\left|\Delta f_{x_0}\right| \le \mathrm{Min}\left\{\lambda^{-1}\left(1-e^{-\lambda}\right), \frac{1}{x_0}\right\}$ where $x_0 \ne 0$

Proof

$$\begin{aligned}
f_0(1) &= (x-1)!\,\lambda^{-x}e^{\lambda}\left[\wp_\lambda h_{(0)}\wp_\lambda\left(1-h_{c_0}\right)\right]\\
&= (x-1)!\,\lambda^{-x}e^{\lambda}\left[e^{-\lambda}\left(1-e^{-\lambda}\right)\right]\\
&\le (x-1)!\,\lambda^{-x}e^{\lambda}\left[e^{-\lambda}-e^{-2\lambda}\right]\\
&\le (x-1)!\,\lambda^{-1}\left(1-e^{-\lambda}\right)\\
&= (x-1)!\,\lambda^{-1}\left[1-1-\frac{\lambda^1}{1!}+\frac{\lambda^2}{2!}-\frac{\lambda^3}{3!}+\cdots\ldots\right]\\
&= \frac{(x-1)!}{1!}\lambda^{-1}\left[-\frac{\lambda^1}{1!}+\frac{\lambda^2}{2!}+\cdots\ldots.\right]\\
&= \frac{(x-1)!}{x_0!}\lambda^{1}\left[\frac{-\lambda^1}{1!}+\frac{\lambda^2}{2!}-\frac{\lambda^3}{3!}+\cdots.\right]
\end{aligned}$$

$\le \frac{\lambda^{-1}}{x_0}\left(1-e^{-\lambda}\right)$.

And $|\Delta f_{x_0}(x)| \le |f_{x_0}(1) = \frac{\lambda^{-1}}{x_0}\left(1-e^{-\lambda}\right)$ **

Combing lemmas 2.1 and (2.2), we obtain $\left|\Delta f_{x_0}\right| \le \mathrm{Min}\left\{\lambda^{-1}\left(1-e^{-\lambda}\right), \frac{1}{x_0}\right\}$ with $x_0 \ne 0$.

Lemma 2.3: For $\lambda > 0$ and $x_0 \in \mathbb{N}\cup\{0\}, x \in \mathbb{N}$ with $\Delta f_{x_0}(x) = f_{x_0}(x+1) - f_{x_0}(x)$ then the following holds

$\left|\Delta f_{x_0}(x)\right| \le \mathrm{Min}\left\{1, \frac{1}{\lambda}\right\}$. (1 9)

By lemma 2.1, we have $\left|\Delta f_{x_0}(x)\right| \le \lambda^{-1}\left(1-e^{-\lambda}\right)$. Clearly $1-e^{-\lambda} < 1$, and also $1-\lambda < e^{-\lambda} \Longrightarrow 1-e^{-\lambda} < \lambda$.

Therefore $\left(1-e^{-\lambda}\right) \le min(1,\lambda)$. Multiplying both sided by $\frac{1}{\lambda}$, we obtain $\lambda^{-1}\left(1-e^{-\lambda}\right) \le min\left(1,\frac{1}{\lambda}\right)$.

This implies that $\left|\Delta f_{x_0}(x)\right| \le \lambda^{-1}\left(1-e^{-\lambda}\right) \le min\left(1,\frac{1}{\lambda}\right)$.Thus

$\left|\Delta f_{x_0}(x)\right| \le min\left(1,\frac{1}{\lambda}\right)$. (20)

Lemma 2.3: let $x_0 \in \mathbb{N}\cup\{0\}$ and $x \in \mathbb{N}$ then we have

$\left|\Delta f_{x_0}(x)\right| \le \lambda^{-2}\left(\lambda + e^{-\lambda} - 1\right)$ (Teerapabolarn and Neammanee)[8]

Lemma 2.4 If $h_\Omega(x)$ is an indicator function of a random variable for event $\Omega = \{x_0\}$, then

$E[h_\Omega(x)] = P_x(x_0)$, (21)

where $x_0 \in \mathbb{N}\cup(0)$.

Proof : By definition $h_\Omega(x) = \begin{cases} 1 & if x \epsilon \Omega \\ 0 & if x \notin \Omega \end{cases}$

$$E[h_\Omega(x)] = \sum_{x=0}^{1} xP(h_\Omega(x) = x) = 0 \times P(h_\Omega(x) = 0) + 1 \times P(h_\Omega(x) = 1) = P(h_\Omega(x) = 1)$$

$= P(\Omega) = P_x(x_0)$.

III. FUNCTIONS

ω-functions were studied and used by many authors. Among others, Papathanassiou and Cacoullos[6] defined a ω function associated with non-negative integer valued random variable in the relation

$\omega(x)\sigma^2 = \frac{1}{p(x)}\sum_{k=0}^{x}[\mu - h(x)]p(x)$ For $x\epsilon S(X)$. (22)

Lemma 3.1 (Egege et al [11]): For $P_\chi(x) > 0$ and $\forall x\epsilon S(X) \cup \{0\}$ there exist a function ω-function such that

$\omega(0) = {}^{\mu}/_{\sigma^2}$. (23)

$\omega(x) = \omega(x-1)\, {}^{P(x-1)}/_{P(x)} + {}^{\mu - h(x)}/_{\sigma^2}$ (24)

Lemma 3.2: Let$p_\chi(x) > 0$, $\forall x \in S(X)$ there exists $\omega(x)$ associated with a random variable X such that

$\frac{p(x-1)}{p(x)} = \frac{x(b+(n-x)c)}{(n-x+1)(r+(x-1)c)}$. (25)

Lemma 3.3: Let $\omega(x)$ be the ω function associated with the Pólya random variable X and $p_\chi(x) > 0$ for every $0 \le x \le n$. Then

$\omega(x) = \frac{(n-x)(r+cx)}{(r+b)\sigma^2}$ (26)

Proof

From (15) For $x > 0$,

$$\omega(x) = \omega(x-1)\frac{p(x-1)}{p(x)} + \frac{\mu - h(x)}{\sigma^2}$$

$\omega(x) = \omega(x-1)\frac{p(x-1)}{p(x)} + \frac{\mu - x}{\sigma^2} = \frac{\mu}{\sigma^2} + \frac{p(x-1)}{p(x)}\omega(x-1) - \frac{x}{\sigma^2}$.

And with $\omega(x-1) = \frac{(n-x+1)(r+(x-1)c)}{(r+b)\sigma^2}$ we obtained $\omega(x) = \frac{(r+cx)(n-x)}{(r+b)\sigma^2}$, which holds.

Proposition 3.1: If a non-negativeinteger valued random variable X have $p_\chi(x) > 0$ for every x in the support of$X$ and finite variance $0 < \sigma^2 < \infty$,then

$E[(X-\mu)f(X)] = \sigma^2 E[\omega(X)\Delta f(X)] < \infty$, (27)

For any function $f: \mathbb{N} \cup \{0\} \to \mathbb{R}$ for which $E\,|\omega(X)\Delta f(X)| < \infty$,where$\Delta f(x) = f(x+1) - f(x)$.

For$f(x) = x$, we have that$E[\omega(X)] = 1$. Papathanassiou and Cacoullos [6].

IV. MAIN RESULTS

Theorem 4.1: Let $0 < d < 1 + r < u$, $\frac{\check{r}}{\check{r}+\breve{b}} = \frac{1+r-d}{u-d}$ and $\frac{\breve{b}}{\check{r}+\breve{b}} = \frac{u-1-r}{u-d}$, let X_0 be the wealth at time zero, and Δ_0 share of stock , for $X_0 - \Delta_0 S_0$ invested in the money market at time zero it worth

$X_0 = \Delta_0 S_0 + (X_0 - \Delta_0 S_0) = c_0$, (28)

such that at time one it worth

$X_1 = \Delta_0 S_1 + (1+r)(X_0 - \Delta_0 S_0) = C_1$, (29)

But $\Delta_0 = \frac{[C_u - C_d]}{(u-d)S_1}$ and a unique solution exist. Then

$X_{t\in[0,1]} = C_{t\in[0,1]}$. (30)

Proof: Given that

$X_1 = \Delta_0 S_1 + (1+r)(X_0 - \Delta_0 S_0) = C_1$,

$= (1+r)X_0 + \Delta_0(S_1 - (1+r)S_0) = C_1$.(31)

Then possible cash position at time $t = 1$ is

$X_0 + \Delta_0\left(\frac{1}{1+r}uS_1 - S_0\right) = \frac{1}{1+r}C_u$ (32)

and

$X_0 + \Delta_0\left(\frac{1}{1+r}dS_1 - S_0\right) = \frac{1}{1+r}C_d$. (33)

Solving equation (32) and (33) simultaneously we obtain

$$\Delta_0\left[\frac{1}{1+r}uS_1 - S_0\right] = \frac{1}{1+r}C_u - X_0$$

$$\Delta_0\left[\frac{1}{1+r}S_1(T) - S_0\right] = \frac{1}{1+r}C_d - X_0$$

$$\Delta_0\left[\frac{1}{1+r}uS_1 - S_0 - \frac{1}{1+r}dS_1 - S_0\right] = \frac{1}{1+r}[C_u - X_0 - C_d - X_0]$$

$$\Delta_0\left[\frac{1}{1+r}uS_1 - S_0 - \frac{1}{1+r}dS_1 + S_0\right] = \frac{1}{1+r}[C_u - X_0 - C_d + X_0]$$
$$\Delta_0\left[\frac{1}{1+r}uS_1 - \frac{1}{1+r}dS_1\right] = \frac{1}{1+r}[C_u - C_d]$$
$$\Delta_0\frac{1}{1+r}[uS_1 - dS_1] = \frac{1}{1+r}[C_u - C_d]$$

Making Δ_0 the subject;

$\Delta_0 = \frac{\frac{1}{1+r}[C_u - C_d]}{\frac{1}{1+r}[uS_1 - dS_1]} = \frac{[C_u - C_d]}{(u-d)S_1}$.

The uniqueness result can be also obtain by recasting the (29) and (30) in terms of matrices. Thus

$\underbrace{\begin{bmatrix} 1 & \frac{1}{1+r}uS_1 \\ 1 & \frac{1}{1+r}dS_1 \end{bmatrix}\begin{bmatrix} X_0 \\ \Delta_0 \end{bmatrix} = \begin{bmatrix} \frac{1}{1+r}C_u \\ \frac{1}{1+r}C_d \end{bmatrix}}_{Ax=b}$.

Uniqueness solution exist if $det(A) \neq 0$

$$det(A) = \left(\frac{1}{1+r}dS_1 - S_0\right) - \left(\frac{1}{1+r}uS_1 - S_0\right) = \frac{1}{1+r}[dS_1 - uS_1] \neq 0$$

From (29) and (30)

$X_0 + \Delta_0\left(\frac{1}{1+r}uS_1 - S_0\right) = \frac{1}{1+r}C_u$ and $X_0 + \Delta_0\left(\frac{1}{1+r}dS_1 - S_0\right) = \frac{1}{1+r}C_d$.

Adding we obtained

$$X_0 + \Delta_0\frac{1}{1+r}[uS_1 - S_0] + \frac{1}{1+r}[dS_1 - S_0] = \frac{1}{1+r}[C_u + C_d]$$

$X_0 + \Delta_0[(uS_1 - dS_1) - S_0] = \frac{1}{1+r}[C_u - C_d]$.

Multiplying uS_1 with $\frac{\check{r}}{\check{r}+\check{b}}$ and dS_1 with $\frac{\check{b}}{\check{r}+\check{b}}$, we obtained

$X_0 + \Delta_0\left[\frac{\check{r}}{\check{r}+\check{b}}uS_1 + \frac{\check{b}}{\check{r}+\check{b}}dS_1\right] - S_0 = \frac{1}{1+r}\left[\frac{\check{r}}{\check{r}+\check{b}}C_u - \frac{\check{b}}{\check{r}+\check{b}}C_d\right]$.

By lemma 3.2 $S_0 = \frac{1}{1+r}\left[\frac{\check{r}}{\check{r}+\check{b}}uS_1 + \frac{\check{b}}{\check{r}+\check{b}}dS_1\right]$, so we have

$X_0 = \frac{1}{1+r}\left[\frac{\check{r}}{\check{r}+\check{b}}C_u - \frac{\check{b}}{\check{r}+\check{b}}C_d\right]$. (34)

Thus $C_0 = X_0$ by (9) and

$C_0 = \frac{1}{1+r}\left[\frac{\check{r}}{\check{r}+\check{b}}C_u - \frac{\check{b}}{\check{r}+\check{b}}C_d\right]$. (35)

V. THE MULTIPLE-PERIODPóLYA MODEL

We started with the one-period model to a multiple-period model, where we assumed that the initial stock price $S_{(0)}$ can increase by a factor u and decrease by a factor d at time one. After oneperiod the stock price will either be $uS_{(0)}$ or $dS_{(0)}$. The stock price can once again go up by $\boldsymbol{u}$ or down by $\boldsymbol{d}$ withpossible prices $\boldsymbol{uuS_{(0)}}$,or $\boldsymbol{u^2S_{(0)}}$,$\boldsymbol{udS_{(0)}}$ and $\boldsymbol{d^2S_{(0)}}$or $\boldsymbol{ddS_{(0)}}$.Now we are interested in the case where there is more than one period for the option to expire$(T = 2)$

Theorem5.1: For a recursively backward sequence with respect to time $t = T-1, T-2 \dots\dots 0,1$

$V_{t+1} = \Delta_t S_{t+1} + 1 + r(C_t - \Delta_t S_t) = C_{t+1}$. (36)

That depend on the subset of $\Omega = \{H, T\}$,with unfair probability and no arbitrage principle such that $C_{t+1} = V_{t+1}$. Then

$C_{(0)} = \frac{1}{(1+r)^n}\sum_{x=0}^{n}\binom{n}{x}\frac{\left(\check{r}/_{\check{r}+\check{b}}\right)^x\left(\check{b}/_{\check{r}+\check{b}}\right)^{n-x}}{\left(\check{r}/_{\check{r}+\check{b}}+\frac{\check{b}}{\check{r}+\check{b}}\right)^n}\max\left[u^x d^{n-x}S_{(0)} - K\right]$. (37)

Proof

For no arbitrage principle $C_{t+1} = V_{t+1}$With$t = T-1, T-2$..Then $C_{(t+1)} = V_{t+1}$,thus

$C_{t+1} = C_2 = \begin{cases} C_{uu} = C_2(HH) \\ C_{ud} = C_2(HT) \\ C_{du} = C_2(TH) \\ C_{dd} = C_2(TT) \end{cases}$ (38)

This implies that

$V_2 = \begin{cases} \Delta_1 S_2(HH) + 1 + r[C_1(H) - \Delta_1 S_1(H)] = C_2(HH) \\ \Delta_1 S_2(HT) + 1 + r[C_1(H) - \Delta_1 S_1(H)] = C_2(HT) \\ \Delta_t S_2(TH) + 1 + r[C_1(T) - \Delta_1 S_1(T)] = C_2(TH) \\ \Delta_1 S_2(TT) + 1 + r[C_1(T) - \Delta_1 S_1(T)] = C_2(TT) \end{cases}$. (39)

Following Osu et al[2] we obtain

$\Delta_1 = \frac{C_2(HH)-C_2(HT)}{S_2(HH)-S_2(HT)} = \frac{C_{uu}-C_{ud}}{uS_1(H)-dS_1(H)} = \frac{C_{uu}-C_{ud}}{(u-d)S_1(H)}$. (40)

Δ_1 is the hedging formula.

Substitute (40)in (39) gives

$\Delta_1 uS_1(H) + 1 + rC_u - 1 + rS_1(H) = C_{uu}$

$\Rightarrow 1 + rC_u + \frac{C_{uu}-C_{ud}\,S_1(H)[u-1+r]}{(u-d)S_1(H)} = C_{uu}$

$\Rightarrow 1 + rC_u + \frac{C_{uu}[u-1+r]}{(u-d)} - \frac{C_{ud}[u-1+r]}{(u-d)} = C_{uu} \Rightarrow 1 + rC_u \frac{\breve{b}}{\check{r}+\breve{b}} C_{uu} - \frac{\check{r}}{\check{r}+\breve{b}} C_{ud} = C_{uu}$

$\Rightarrow 1 + rC_u = C_{uu} - \frac{\breve{b}}{\check{r}+\breve{b}} C_{uu} + \frac{\breve{b}}{\check{r}+\breve{b}} C_{ud} = \left(1 - \frac{\breve{b}}{\check{r}+\breve{b}}\right) C_{uu} + \frac{\breve{b}}{\check{r}+\breve{b}} C_{ud}$. So that

$C_u = \frac{1}{1+r}\left[\frac{\check{r}}{\check{r}+\breve{b}} C_{uu} + \frac{\breve{b}}{\check{r}+\breve{b}} C_{ud}\right]$ (41)

Now solving also equations (39) we obtain the value of a_1 and $C_1(T) = C_d$ as

$\Delta_1 = \frac{C_2(HT)-C_2(TT)}{S_2(TH)-S_2(TT)} = \frac{C_{ud}-C_{dd}}{uS_1(T)-dS_1(T)}$. (42)

And Substituting Δ_1 into (40) we have

$C_d = \frac{1}{1+r}\left[\frac{\check{r}}{\check{r}+\breve{b}} C_{ud} + \frac{\breve{b}}{\check{r}+\breve{b}} C_{dd}\right]$. (43)

Where $\frac{\check{r}}{\check{r}+\breve{b}} = \frac{1+r-d}{u-d}$ and $\frac{\breve{b}}{\check{r}+\breve{b}} = \frac{u-1-r}{u-d}$ which are called risk neutral probabilities.

Substituting equations (42 and (43) into equation (35) we have

$C_{(0)} = \frac{1}{1+r}\left[\frac{\check{r}}{\check{r}+\breve{b}}\left(\frac{1}{1+r}\right)\frac{\check{r}}{\check{r}+\breve{b}} C_{uu} + \frac{\breve{b}}{\check{r}+\breve{b}} C_{ud} + \frac{\breve{b}}{\check{r}+\breve{b}}\left(\frac{1}{1+r}\right)\frac{\check{r}}{\check{r}+\breve{b}} C_{ud} + \frac{\breve{b}}{\check{r}+\breve{b}} C_{dd}\right]$.

Or better of

$$C_{(0)} = \frac{1}{1+r}\left[\frac{1}{1+r}\left(\frac{\check{r}}{\check{r}+\breve{b}}\right)^2 C_{uu} + 2\frac{\check{r}}{\check{r}+\breve{b}}\frac{\breve{b}}{\check{r}+\breve{b}} C_{ud} + \left(\frac{\breve{b}}{\check{r}+\breve{b}}\right)^2 C_{dd}\right]$$

$$= \frac{1}{1+r^2}\left[\left(\frac{\check{r}}{\check{r}+\breve{b}}\right)^2 C_{uu} + 2\frac{\check{r}}{\check{r}+\breve{b}}\frac{\breve{b}}{\check{r}+\breve{b}} C_{ud} + \left(\frac{\breve{b}}{\check{r}+\breve{b}}\right)^2 C_{dd}\right]. \quad (44)$$

Where $C_{uu}\, C_{ud}$ *and* C_{dd} is the payoff of the stock at $t = 2$. The pay-off values can be generalized of the form $C_T(x) = Max\,[u^x d^{n-x} S_{(0)} - K, 0]$. Where $x = 0,1,2 \ldots, n$ and x depends on the factor u.

Now $C_{(0)} = \frac{1}{1+r^2}\left[\left(\frac{\check{r}}{\check{r}+\breve{b}}\right)^2 C_{uu} + 2\frac{\check{r}}{\check{r}+\breve{b}} 1 - \frac{\check{r}}{\check{r}+\breve{b}} C_{ud} + \left(1 - \frac{\check{r}}{\check{r}+\breve{b}}\right)^2 C_{dd}\right] \frac{1}{(1+r)^2}\left[\left(\frac{\check{r}}{\check{r}+\breve{b}}\right)^2 C_2(2) + 2\frac{\check{r}}{\check{r}+\breve{b}} 1 -\right.$
rr+bC21+br+b2C20

$$= \frac{1}{(1+r)^2}\left[\left(\frac{\check{r}}{\check{r}+\breve{b}}\right)^2 + 2\frac{\check{r}}{\check{r}+\breve{b}} 1 - \frac{\check{r}}{\check{r}+\breve{b}} + \left(1 - \frac{\check{r}}{\check{r}+\breve{b}}\right)^2\right] Max\,[u^x d^{n-x} S_{(0)} - K, 0].$$

$$C_{(0)} = \frac{1}{(1+r)^2}\left[\left(\frac{\check{r}}{\check{r}+\breve{b}}\right)^2 + 2\frac{\check{r}}{\check{r}+\breve{b}} 1 - \frac{\check{r}}{\check{r}+\breve{b}} + \left(\frac{\breve{b}}{\check{r}+\breve{b}}\right)^2\right] Max\,[u^x d^{n-x} S_{(0)} - K, 0]$$

$$= \frac{1}{(1+r)^2}\left(\frac{\check{r}}{\check{r}+\breve{b}} + \frac{\breve{b}}{\check{r}+\breve{b}}\right)^2 Max[u^x d^{n-x} S_{(0)} - K, 0].$$

By Binomial theory expansion, we have that

$$C_{(0)} = \frac{1}{(1+r)^2}\left(\frac{\check{r}}{\check{r}+\breve{b}} + \frac{\breve{b}}{\check{r}+\breve{b}}\right)^2 \max[u^x d^{n-x} S_{(0)} - K]$$

$$= \frac{1}{(1+r)^2}\sum_{x=0}^{n}\binom{2}{x}\frac{\left(\check{r}/_{\check{r}+\breve{b}}\right)^x \left(\breve{b}/_{\check{r}+\breve{b}}\right)^{2-x}}{\left(\check{r}/_{\check{r}+\breve{b}} + \frac{\breve{b}}{\check{r}+\breve{b}}\right)^2} \max[u^x d^{n-x} S_{(0)} - K].$$

Where $x = 0,1, \ldots n$.

Generally

$$C_{(0)} = \frac{1}{(1+r)^n}\sum_{x=0}^{n}\binom{n}{x}\frac{\left(\check{r}/_{\check{r}+\breve{b}}\right)^x \left(\breve{b}/_{\check{r}+\breve{b}}\right)^{n-x}}{\left(\check{r}/_{\check{r}+\breve{b}} + \frac{\breve{b}}{\check{r}+\breve{b}}\right)^n} \max[u^x d^{n-x} S_{(0)} - K].$$

Theorem 5.2: .Let X be the Pólya random variable with $\lambda = \frac{rn}{r+b}$, and $(n-1)c \leq r$ for $c > 0\ \forall x_0 \in \{0,1 \ldots \ldots n\}$,the following exist

$|P_\chi(x_0) - \wp_\lambda(x_0)| \leq \lambda^{-1}(\lambda + e^{-\lambda} - 1)\frac{r(r+b+c)-b(n-1)c}{(r+b)(r+b+c)}$. (45)

For $x_0 \in \{1, \ldots\ldots, n\}$

$|P_\chi(x_0) - \wp_\lambda(x_0)| \leq Min\left(1 - e^{-\lambda}, \frac{\lambda}{x_0}\right)\frac{r(r+b+c)-b(n-1)c}{(r+b)(r+b+c)}$, (46)

and

$|P_\chi(x_0) - \wp_\lambda(x_0)| \quad \leq \min\left(\frac{r(r+b+c)-b(n-1)c}{(r+b)(r+b+c)}, \lambda\frac{r(r+b+c)-b(n-1)c}{(r+b)(r+b+c)}\right)$ (47)
for$\forall x_0 \in \mathbb{N} \cup (0)$.

Proof: Substituting $x$by X from on light hand side (6) and taking expectation, we have

$$E[h_\Omega(x) - \wp_\lambda(h)] = E[\lambda f(x+1) - xf(x)]$$
$$p_\chi(x_0) - \wp_\lambda(x_0) = E\lambda[f(X+1)] - E[Xf(X)]$$
$$= \lambda E[f(X+1) - f(X) + f(X)] - E[Xf(X)]$$
$$= E[\Delta f(X) + f(X)] - E[Xf(X)]$$
$$= \lambda E[\Delta f(X) + f(X)] - E[(X-\mu)f(X)] - E[\mu f(X)]$$

$= E[\lambda\Delta f(X) + \lambda f(X)] - E[(\lambda-\mu)f(X)] - E[(X-\mu)f(X)]$. (48)

Recall from Proposition 3.1 that

$$E[(X-\mu)f(X)] = \sigma^2 E[\omega(X)\Delta f(X)]$$
$$= \lambda E[\Delta f(X)] + E[\lambda-\mu]f(X) - \sigma^2 E[\omega(X)\Delta f(X)]$$
$$= E\left[\left(\lambda - \sigma^2\omega(X)\right)\Delta f(X)\right] + E[(\lambda-\mu)]f(X)$$

$|p_\chi(x_0) - \wp_\lambda(x_0)| = |E\left[\left(\lambda - \sigma^2\omega(X)\right)\Delta f(X)\right] + E[(\lambda-\mu)]f(X)|$

$\leq [|\lambda - \sigma^2\omega(X)|\Delta f(X)|] + |\lambda-\mu|E|f(X)| \leq \sup_{x\geq 1} |\Delta f(X)|E|\lambda - \sigma^2\omega(X)| + |\lambda-\mu|E|f(X)|$.

Since λ=μ , for $x \in \mathbb{N}$ we have

$\lambda - \sigma^2\omega(X) = \frac{rn}{(r+b)} - \frac{(n-x)(r+cx)}{(r+b)} = \frac{[(n-x)c-r]x}{(r+b)} \leq 0$.

And $|\lambda - \sigma^2\omega(X)| \geq 0$, $E|\lambda - \sigma^2\omega(X)| = \lambda - \sigma^2 E[\omega(X)]$, where $E[\omega(X)] = 1$.

$$E|\lambda - \sigma^2\omega(X)| = \lambda - \sigma^2 = \frac{rn}{(r+b)} - \frac{nr(r+b+cn)(r+b-r)}{(r+b)^2(r+b+c)}$$
$$= \frac{nr}{(r+b)} - \frac{nrb\,(r+b+cn)}{(r+b)^2(r+b+c)} = \frac{nr\,[(r+b)(r+b+c) - nrb\,(r+b+cn)]}{(r+b)^2(r+b+c)}$$
$$= \frac{nr\,[(r+b)(r+b+c) - b(r+b+cn)]}{(r+b)^2(r+b+c)} = \frac{nr\left[(r+b)(r+b+c) - b[r+b+cn-c+c]\right]}{(r+b)^2(r+b+c)}$$
$$= \frac{nr\left[(r+b)(r+b+c) - b[(r+b+c)+(cn-c)]\right]}{(r+b)^2(r+b+c)}$$
$$= \frac{nr[(r+b)(r+b+c) - b(r+b+c) - b(cn-c)]}{(r+b)^2(r+b+c)}$$
$$= \frac{nr}{(r+b)}\frac{[(r+b)^2(r+b+c) - b(n-1)c - b(r+b+c)]}{(r+b)^2(r+b+c)}$$
$$= \frac{nr}{(r+b)}\frac{(r+b)^2(r+b+c) - \frac{b}{(r+b)}(n-1)c - \frac{b}{(r+b)}(r+b+c)}{(r+b+c)(r+b)^2}$$
$$= \lambda\frac{(r+b)^2(r+b+c) - \frac{b(n-c)c}{(r+b)} - \frac{b}{(r+b)}(r+b+c)}{(r+b)^2(r+b+c)}$$
$$= \lambda(r+b)\frac{[(r+b)(r+b+c) - b(r+b+c) - b(n-1)c]}{(r+b)^2(r+b+c)}.$$
$$\lambda(r+b)\frac{[(r+b)(r+b+c) - b(r+b+c) - b(n-1)c]}{(r+b)(r+b+c)} = \lambda\frac{[-b(n-1)c + [r+b+c](r+b-b)]}{(r+b)(r+b+c)}$$
$$\lambda\frac{[-b(n-1)c + r(r+b+c)]}{(r+b)(r+b+c)} = \lambda\frac{r(r+b+c) - b(n-1)c}{(r+b)(r+b+c)}.$$
$$E|\lambda - \sigma^2\omega(X)| = \lambda\frac{r(r+b+c) - b(n-1)c}{(r+b)(r+b+c)} \Rightarrow \left|p_\chi(x_0) - \wp_\lambda(x_0)\right| \leq \sup_{x\geq 1} |\Delta f(X)|E|\lambda - \sigma^2\omega(X)|$$

$\leq \sup_{x\geq 1} |\Delta f(X)|\, \lambda\frac{r(r+b+c)-b(n-1)c}{(r+b)(r+b+c)}$ (By lemma 2.2 , 2.3 and 2.4). The theorem is proved.

In general we have

$$\left|p_\chi(x_0) - \wp_\lambda(x_0)\right| \leq \lambda^{-1}\left(\lambda + e^{-\lambda} - 1\right)\frac{r(r+b+c) - b(n-1)c}{(r+b)(r+b+c)}\, x_0 = 0$$
$$\leq \min\left\{\left(1 - e^{-\lambda}\right)\frac{r(r+b+c) - b(n-1)c}{(r+b)(r+b+c)}, \frac{\lambda}{x_0}\frac{r(r+b+c) - b(n-1)c}{(r+b)(r+b+c)}\right\} 0 < x_0 \leq n$$
$$\leq \min\left(\lambda\frac{r(r+b+c) - b(n-1)c}{(r+b)(r+b+c)}, \frac{r(r+b+c) - b(n-1)c}{(r+b)(r+b+c)}\right)$$

for $\forall x_0 \in \mathbb{N} \cup (0)$.

This can be written as

$$|p_\chi(x_0) - \wp_\lambda(x_0)| \leq \begin{cases} \lambda^{-1}(\lambda + e^{-\lambda} - 1)\frac{r(r+b+c)-b(n-1)c}{(r+b)(r+b+c)} \; for\; x_0 = 0 \\ \min\left\{(1-e^{-\lambda})\frac{r(r+b+c)-b(n-1)c}{(r+b)(r+b+c)}, \frac{\lambda}{x_0}\frac{r(r+b+c)-b(n-1)c}{(r+b)(r+b+c)}\right\} 0 < x_0 \leq n. \\ \min\left(\lambda\frac{r(r+b+c)-b(n-1)c}{(r+b)(r+b+c)}, \frac{r(r+b+c)-b(n-1)c}{(r+b)(r+b+c)}\right) x_0 = 0,1,\dots n \end{cases} \quad (49)$$

Corollary 5.1 the following true

1. $\lambda^{-1}(\lambda + e^{-\lambda} - 1) < (1 - e^{-\lambda})$
2. $\min\left\{1 - e^{-\lambda}, \frac{\lambda}{x_0}\right\} \leq (1 - e^{-\lambda})$
3. $(1 - e^{-\lambda}) \leq \; min(1, \lambda)$

Remark: With the above corollary it is clear that non-uniform is sufficient enough to for accuracy in approximation.

VI. NUMERICAL EXAMPLES

Using the same numerical examples in Samson et al[11], and Bright Osu et al [2]to illustrate how well Pólya can be approximated by Poisson in terms of point metric Pólya distribution can be associated with finance terms to evaluate the call option respectively.

Example 6.1

Suppose that $n = 5, r = 10, \; (r + b) = 1000, c = 1, \lambda = 0.05$ then for
Uniform Bound;

$$|P_\chi(x) - \wp_\lambda(x)| \quad \leq 0.000294767 \;\; x = 0.1 \dots\dots n$$

and for Non-uniform Bound

$$|p_\chi(x_0) - \wp_\lambda(x_0)| \leq \begin{cases} 0.00014861 \;\; if \;\; x_0 = 0 \\ 0.000294767 \,, if \; x_0 = 1,2,3 \\ \frac{0.05}{x_0}(0.00643956) \; if \; x_0 = 4 \dots n \\ \min\{0.00030220, 0.00604396\} x_0 = 1,2.n \end{cases} .$$

Example 6.2

Suppose that $= 10, r = 10$,$(r + b) = 1000, \lambda \; = 0.1$ and $c = 1$, then for Uniform Bound

$$|P_\chi(x) - \wp_\lambda(x)| \quad \leq 0.000104574 \;\; x = 0,1 \dots\dots n$$

and Non –uniform Bound

$$|p_\chi(x_0) - \wp_\lambda(x_0)| \leq \begin{cases} 0.000053123 \;\; if \;\; x_0 = 0 \\ 0.0000104574 \,, if \; x_0 = 1,2,3 \\ \frac{0.1}{x_0}(0.001098901) x_0 \; = 4 \dots\dots n \\ \min\{0.00010989, \;\; 0.001098901\} x_0 = 0,1,2..n \end{cases} .$$

Example 6.3

Suppose that $n = 20$,$r = 25$,$(r + b) = 1000$ and $\lambda = 0.5$ and $c = 1$, for Uniform Bound

$$|P_\chi(x) - \wp_\lambda(x)| \quad \leq 0.002545169 \;\; x = 0,1 \dots\dots n$$

and Non-uniform Bound

$$|p_\chi(x_0) - \wp_\lambda(x_0)| \leq \begin{cases} 0.001378258 \;\; if \;\; x_0 = 0 \\ 0.002545169 \,, if \; x_0 = 1,2,3,4 \\ \frac{0.5}{x_0}(0.006493506) x_0 \; 5 \dots\dots n \\ \min\{0.003246753, 0.006493506\} x_0 = 0,1 \dots n \end{cases} .$$

Example 6.4

Suppose that $n = 25$, $r = 25$, $(r + b) = 1000$,$\lambda = 0.625$ and $c = 1$, for Uniform Bound

$$|P_\chi(x) - \wp_\lambda(x)| \quad \leq 0.00754441 \;\; x = 0,1 \dots\dots n$$

and for Non-uniform Bound

$$|p_\chi(x_0) - \wp_\lambda(x_0)| \leq \begin{cases} 0.004162264 \;\; if \;\; x_0 = 0 \\ 0.00754441 \,, if \; x_0 = 1,2,3,4 \\ \frac{0.625}{x_0}(0.001623377) x_0 \; 5 \dots\dots n \\ \min\{0.001014610, 0.001623377\} x_0 = 0,1..n \end{cases} .$$

Example 6.5

Suppose that $n = 25$, $r = 35$, $(r + b) = 1000$, $\lambda = 0.875$, and $c = 1$. For Uniform Bound

$$|P_{\chi}(x) - \wp_{\lambda}(x)| \quad \leq 0.006897456 \;\; x = 0,1 \dots \dots n$$

and for Non- uniform Bound

$$|p_{\chi}(x_0) - \wp_{\lambda}(x_0)| \leq \begin{cases} 0.003945365 \;\; if \;\; x_0 = 0 \\ 0.006897456 \,, if \; x_0 = 1,2,3,4 \\ \frac{0.875}{x_0}(0.011828172)x_0 \; 5 \dots \dots n \\ \min 0.010380245, 0.011863137 \; x_0 = 0,1..n \end{cases}.$$

For Option pricing via Pólya distribution, we obtain

Example 6.6: Let$S_{(0)} = 100$, $K = 100$, $u = 1.2$, $d = 0.8$ $r = 10\%$ and $1 + r = 1 + 10\%$

$\frac{\check{r}}{\check{r}+\breve{b}} = \frac{R-d}{u-d}\frac{1.1-0.8}{1.2-0.8} = \frac{75}{100}and\frac{\breve{b}}{\check{r}+\breve{b}} = 1 - \frac{75}{100} = \frac{25}{100}$.

Then the possible ending values for the call option after T =2 are

$C_T(x)$ = Max $[u^x d^{n-x} S(0) - K, 0]$ where$x = 0,1,2..n$, then

$C_2(2) = C_{uu} = Max[1.2^2 \times 0.8^{2-2} \times 100 - 100{,}0] = 44$

$C_2(1) = C_{ud} = Max[1.2^1 \times 0.8^{2-1} \times 100 - 100{,}0] = 0$

$C_2(0) = C_{dd} = Max[1.2^0 \times 0.8^{2-0} \times 100 - 100{,}0] = 0$,

$$C_{(0)} = \frac{1}{1.1^2}\left[\begin{array}{c} \frac{2!}{0!2!}\frac{(0.75)^0\times(0.25)^2}{\left(\frac{75}{100}+\frac{25}{100}\right)^2}\times 0 + \frac{2!}{1!1!}\frac{(0.75)^1\times(0.25)^1}{\left(\frac{75}{100}+\frac{25}{100}\right)^2}\times 0 + \\ \frac{2!}{2!0!}\frac{(0.75)^2\times(0.25)^0}{\left(\frac{75}{100}+\frac{25}{100}\right)^2}\times 44 \end{array}\right] = \frac{1}{1.21}[1 \times 0.562 \times 1 \times 44] = \$20.45.$$

Example 6.7: Now assume that $S_{(0)} = 100$, $K = 100, r = 7\%$, $T = 3$, $u = 1.1$ and $d = 0.9$ in addition to above example. Given

$$C_{(0)} = \frac{1}{(1+r)^n}\sum_{x=0}^{n}\binom{n}{x}\frac{\left(\check{r}/_{\check{r}+\breve{b}}\right)^x\left(\breve{b}/_{\check{r}+\breve{b}}\right)^{n-x}}{\left(\check{r}/_{\check{r}+\breve{b}}+\frac{\breve{b}}{\check{r}+\breve{b}}\right)^n}\max\left[u^x d^{n-x} S_{(0)} - K\right].$$

Then the possible ending values for the call option after T =3.

$C_T(x)$ = Max $[u^x d^{n-x} S(0) - k, 0]$ where $x = 0,1,2..n$

$C_3(3) = Max[1.1^3 \times 0.90^{3-3} \times 100 - 100{,}0] = 33.10$,

$C_3(2) = C_{ud} = Max[1.1^2 \times 0.90^{3-2} \times 100 - 100{,}0] = 8.90$,

$C_3(1) = C_{dd} = Max[1.1^1 \times 0.90^{3-1} \times 100 - 100{,}0] = 0$,

$C_3(0) = C_{dd} = Max[1.1^0 \times 0.90^{3-0} \times 100 - 100{,}0] = 0$.

By above $$C_{(0)} = \frac{1}{(1.07)^3}\left[\begin{array}{c} \frac{3!}{0!3!}\frac{(0.85)^0\times(0.15)^3}{\left(\frac{85}{100}+\frac{15}{100}\right)^3}\times 0 + \frac{3!}{1!2!}\frac{(0.85)^1\times(0.15)^2}{\left(\frac{85}{100}+\frac{15}{100}\right)^3}\times 0 + \\ \frac{3!}{2!1!}\frac{(0.85)^2\times(0.15)^1}{\left(\frac{85}{100}+\frac{15}{100}\right)^3}\times 8.90 + \frac{3!}{3!0!}\frac{(0.85)^3\times(0.15)^0}{\left(\frac{85}{100}+\frac{15}{100}\right)^3}\times 33.10 \end{array}\right] = \$18.96.$$

Example 6.8; Given that$S_{(0)} = 80$, K= 100, $u = 1.2$, $d = 0.8$, $r = 10\%$ and $T = 3$

Now with the above data's we obtain $r + 1 = 1.1$, $\frac{\check{r}}{\check{r}+\breve{b}} = 0.75$ and $\frac{\breve{b}}{\check{r}+\breve{b}} = 0.25$. Then the possible ending values for the call option after $T = 3$.

$C_T(x)$ = Max $[u^x d^{n-x} S(0) - k, 0]$ where $x = 0,1,2..n$

$C_3(3) = Max[1.2^3 \times 0.80^{3-3} \times 80 - 100,0] = 38.40,$

$C_3(2) = C_{ud} = Max[1.2^2 \times 0.80^{3-2} \times 80 - 100,0] = 0,$

$C_3(1) = C_{dd} = Max[1.2^1 \times 0.80^{3-1} \times 80 - 100,0] = 0,$

$$C_{(0)} = \begin{bmatrix} \frac{3!}{0!3!}\frac{(0.75)^0\times(0.25)^3}{\left(\frac{75}{100}+\frac{25}{100}\right)^3} \times 0 + \frac{3!}{1!2!}\frac{(0.75)^1\times(0.25)^2}{\left(\frac{75}{100}+\frac{25}{100}\right)^3} \times 0 + \\ \frac{3!}{2!1!}\frac{(0.75)^2\times(0.25)^1}{\left(\frac{75}{100}+\frac{25}{100}\right)^3} \times 0 + \frac{3!}{3!0!}\frac{(0.75)^3\times(0.25)^0}{\left(\frac{75}{100}+\frac{25}{100}\right)^3} \times 38.40 \end{bmatrix} = \$12.13.$$

From the above numerical results Examples 6.1-6.5, it is found that the results obtained from non-uniform bound are better than uniform bound which is in agreement with Corollary 5.1

1. $\lambda^{-1}(\lambda + e^{-\lambda} - 1) < (1 - e^{-\lambda})$
2. $\min\left\{1 - e^{-\lambda}, \frac{\lambda}{x_0}\right\} \le (1 - e^{-\lambda})$.

When the bound is very small, a good approximation of Pólya to Poisson distribution is obtained. Examples 6.6-6.8 show that Pólya distribution associated with finance term can be used to evaluate the call option.

VII. CONCLUSION

In this work, non-uniform bound is an estimate of the point metric between the Pólya and Poisson distribution. This bound is also a criterion for measuring the accuracy of the approximation of Pólya by Poisson that is if the bound is small,then a good approximation of Pólya to Poisson distribution is obtained. And if the bound is large, the Poisson distribution is not appropriate to approximate the Pólya distribution. For the bound to be small $n \le r$ exist with $\frac{r}{r+b}$ small and$c = 1$.This is in agreement with Egege et al [11]. For a good approximation of Pólya to Poisson the following must hold.

i. $n \le r$ and small
ii. $\frac{r}{r+b}$ is small
iii. λ is small
iv. $r + b$ sufficiently large

Pólya distribution will provide a good approximation to Poisson distribution when $n \ge 5$ and $r \ge 10$ and approximate excellently when $n \le r$**.** If the year for the option to expire or exercised is not continuous; the Pólya distribution can be used to evaluate the price.

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