

## Properties Of Gsp-Separation Axioms In Topology

<sup>1</sup>Govindappa Navalagi And <sup>\*2</sup>R G Charantimath

<sup>1</sup>Department of Mathematics KIT Tiptur 572202 ,Karnataka India

<sup>\*2</sup>Department of Mathematics KIT Tiptur 572202 ,Karnataka India

Corresponding Author: R G Charantimath

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**ABSTRACT:** In this paper we define and study  $gsp$ -separation axioms , namely ,  $gsp-T_0$  ,  $gsp-T_1$  ,  $gsp-T_2$   $gsp-R_0$  and  $gsp-R_1$  spaces using  $gsp$ -open sets due to J.Dontchev (1995). Also, we study the comparison of these  $gsp$ -separation axioms with the existing  $gp$ -separation axioms and  $\alpha g$ -separation axioms . Further , we also introduce and study the notions of  $g^*$ -separations .

**KEY WORDS:** semipreopen sets,  $gsp$ -closed sets ,  $g^*$ -closed sets preopen sets ,  $gs$ -closed sets,  $gsp$ -irresoluteness, and strongly  $g^*$ -continuums functions

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### I. INTRODUCTION

In 1982 A S Mashhour et al[8] have defined and studied the concept of pre-open sets and Spre-continuous functions of topology. In 1983 S.N.Deeb et al [5] have defined and studied the concepts of pre-closed sets , preclosure operator ,  $p$ -regular spaces and pre-closed functions in topology. In 1998 , T.Noiri et al [14] have defined the concepts of  $gp$ -closed sets and  $gp$ -closed functions in topology. In 2012 , Navalagi et al. [12] have defined and studied the concepts of Generalized pre-separation axioms like ,  $gp-T_0$  ,  $gp-T_1$  ,  $gp-T_2$  ,  $gp-R_0$  and  $gp-R_1$  spaces using  $gp$ -open sets due to T.Noiri et al [14] . In 1986, D. Andrijivic [1] introduced and studied the notion of semipre open sets, semipreclosed sets ,semipreinterior operator and semipre-closed operator in topological spaces. In 1965 , Njstad [13] has defined the concept of  $\alpha$ -open sets in topological spaces . In 1983, A.S.Mashhour et al [9] have defined and studied the concepts of  $\alpha$ -closed sets ,  $\alpha$ -closure operator,  $\alpha$ -continuity ,  $\alpha$ -openness and  $\alpha$ -closedness in topology. For the first time , N.Levine [6] has introduced the notion  $g$ -closed sets and  $g$ -open sets in topology. In 1994, H.Maki et al [7] have defined and studied the concepts of  $\alpha g$ -closed sets in topological spaces .

in topological spaces . Recently , in 2014 Thakur C.K.Raman et al [3& 15] have defined and studied the concepts of  $\alpha g$ -separation axioms in tology. In 1995 , J.Dontchev [4] has defined and studied the concept of  $gsp$ -closed sets,  $gsp$ -open sets ,  $gsp$ -continuous functions and  $gsp$ -irresoluteness in topology. In this paper , using  $gsp$ -open sets ,we define and study the notions of  $gsp-T_0$  ,  $gsp-T_1$  ,  $gsp-T_2$   $gsp-R_0$  and  $gsp-R_1$  spaces .

### II. PRELIMINARIES

Throughout this paper  $( X , \tau )$  and  $( Y , \sigma )$  (or simply  $X$  and  $Y$  ) denote topological spaces on which no separation axioms are assumed unless explicitly stated . If  $A$  be a subset of  $X$ , the closure of  $A$  and the interior of  $A$  is denoted by  $Cl( A)$  and  $Int( A )$  ,respectively.

We give the following define are useful in the sequel :

**Definition 2.1:** The subset of  $A$  of  $X$  is said to be.

- (i) A pre-open [8]set, if  $A \subset Int(Cl(A))$
- (ii) A semi-pre open[1] set , if  $A \subset Cl(Int(Cl(A)))$
- (iii)  $\alpha$ -open [13] set, if  $A \subset Int(Cl(Int(A)))$

The compliment of a pre-open (resp., semipre-open ,  $\alpha$ -open) set is called pre-closed [5] (resp., semipre-closed [1],  $\alpha$ -closed[9] ) set in space  $X$  . The family of all pre-open (resp. semipre-open,  $\alpha$ -open) sets of a space  $X$  is denoted by  $PO(X)$  ( resp.,  $SPO(X)$  ,  $\alpha O(X)$  ) and that of pre-closed ( resp.semipre-closed ,  $\alpha$ -closed) sets of a space  $X$  is denoted by  $PF(X)$ , ( resp. $SPF(X)$  ,  $\alpha F(X)$ ).

**Definition 2.2[5 ]** : The intersection of all pre-closed sets of  $X$  containing subset  $A$  is called the pre-closure of  $A$  and is denoted by  $pCl(A)$ .

**Definition 2.3[1]** : The intersection of all semipre-closed sets of  $X$  containing subset  $A$  is called the semipre-closure of  $A$  and is denoted by  $spCl(A)$ .

**Definition 2.4[ 9 ]** : The intersection of all  $\alpha$ -closed sets of  $X$  containing subset  $A$  is called the  $\alpha$ -closure of  $A$  and is denoted by  $\alpha Cl(A)$ .

**Definition 2.5[5]**: The union of all pre-open sets of  $X$  contained in  $A$  is called the pre-interior of  $A$  and is denoted by  $pInt(A)$ .

**Definition 2.6[1]**: The union of all semipre-open sets of  $X$  contained in  $A$  is called the semipre-interior of  $A$  and is denoted by  $spInt(A)$ .

**Definition 2.7[9]**: The union of all  $\alpha$ -open sets of  $X$  contained in  $A$  is called the  $\alpha$ -interior of  $A$  and is denoted by  $\alpha Int(A)$ .

**Definition 2.8** : A sub set  $A$  of a space  $X$  is said to be :

- (i) a generalized closed ( briefly, g- closed ) [6] set if  $Cl(A) \subseteq U$  whenever  $A \subseteq U$  and  $U$  is open set in  $(X, \tau)$
- (ii) a  $\alpha$ - generalized closed ( briefly,  $\alpha g$ - closed ) [7] set if  $\alpha Cl(A) \subseteq U$  whenever  $A \subseteq U$  and  $U$  is open set in  $(X, \tau)$
- (iii) a generalized semi-preclosed ( briefly, gsp- closed ) [4] set if  $spCl(A) \subseteq U$  whenever  $A \subseteq U$  and  $U$  is open in  $(X, \tau)$
- (iv) a generalized pre -closed ( briefly, gp- closed ) [14] set if  $pCl(A) \subseteq U$  whenever  $A \subseteq U$  and  $U$  is open in  $(X, \tau)$

The complement of a g-closed ( resp,  $\alpha g$ -closed, gsp-closed, , gp-closed) set in  $X$  is called g-open ( resp.  $\alpha g$ -open, gsp- open, , gp- open) set in  $X$ . The family of all gsp-open sets of  $X$  is denoted by  $GSPO(X)$ .

**Definition 2.9[12 ]**: The intersection of all gp-closed sets of  $X$  containing subset  $A$  is called the gp-closure of  $A$  and is denoted by  $gpCl(A)$ .

**Definition 2.10[3 ]**: The intersection of all  $\alpha g$ -closed sets of  $X$  containing subset  $A$  is called the  $\alpha g$ -closure of  $A$  and is denoted by  $\alpha gCl(A)$ .

**Definition 2.11 [12]**: A space  $X$  is called generalized pre- $T_1$  (briefly written as  $gp-T_1$ ) iff to each pair of distinct points  $x,y$  of  $X$ , there exists a pair of gp-open sets containing  $x$  but not  $y$  and the other containing  $y$  but not  $x$ .

**Definition 2.12 [12 ]** : A space  $X$  is said to be  $gp-T_2$  space if for each pair of distinct points of  $X$  there exist disjoint gp-open sets containing them.

**Definition 2.13 [3&15 ]**: A space  $X$  is called  $\alpha$ -generalized- $T_0$  (briefly written as  $\alpha g-T_0$ ) iff to each pair of distinct points  $x,y$  of  $X$ , there exists a  $\alpha g$ -open set containing one but not the other.

**Definition 2.14 [3&15 ]**: A space  $X$  is called  $\alpha$ -generalized- $T_1$  (briefly written as  $\alpha g-T_1$ ) iff to each pair of distinct points  $x,y$  of  $X$ , there exists a pair of  $\alpha g$ -open sets containing  $x$  but not  $y$  and the other containing  $y$  but not  $x$ .

**Definition 2.15 [10 ]**: A space  $X$  is called semipre- $T_0$  (briefly written as semipre- $T_0$ ) iff to each pair of distinct points  $x,y$  of  $X$ , there exists a semipre-open set containing one but not the other.

**Definition 2.16 [10 ]**: A space  $X$  is called semipre- $T_1$  (briefly written as semipre- $T_1$ ) iff to each pair of distinct points  $x,y$  of  $X$ , there exists a pair of semipre-open sets containing  $x$  but not  $y$  and the other containing  $y$  but not  $x$ .

### III. PROPERTIES OF GSP-SEPARATION AXIOMS

We, define the following

**Definition 3.1**: A space  $X$  is called  $gsp-T_0$  iff to each pair of distinct points  $x,y$  of  $X$ , there exists a  $gsp$ -open set containing one but not the other .

**Definition 3.2** : A space  $X$  is said to be  $gp-T_0$  space if for each pair of distinct points of  $X$  there exists a  $gp$ -open set containing one but not the other.

Clearly, every semipre- $T_0$  is  $gsp-T_0$  . Also, we have the following ,

**Note 3.3** : In view of definitions of  $\alpha g$ -closed,  $gp$ -closed sets and  $gsp$ -closed sets, the following is observed in [2] :

$$\alpha g\text{-closed set} \Rightarrow gp\text{-closed set} \Rightarrow gsp\text{-closed set}$$

Hence we have the following implication:

$$\alpha g\text{-}T_0\text{-space} \rightarrow gp\text{-}T_0\text{-space} \rightarrow gsp\text{-}T_0\text{-space}$$

We ,define the following

**Definition 3.4 :** A generalized semipre-closure of set  $A$  is denoted by  $\text{gspCl}(A)$ , is the intersection of all gsp-closed sets that contain  $A$

We characterize  $\text{gsp-T}_0$ -spaces in the following

**Theorem 3.5:** If in any topological space  $X$ , gsp-closures of distinct points are distinct, then  $X$  is  $\text{gsp-T}_0$

**Proof:** Let  $x, y \in X$ ,  $x \neq y$  imply  $\text{gspCl}(\{x\}) \neq \text{gspCl}(\{y\})$ . Then there exists a point  $z \in X$  such that  $z$  belongs one of two sets, say,  $\text{gspCl}(\{y\})$  but not to  $\text{gspCl}(\{x\})$ . If we suppose that  $z \in \text{gspCl}(\{x\})$ , then  $z \in \text{gspCl}(\{y\}) \subsetneq z \in \text{gspCl}(\{x\})$ , which is contradiction. So,  $y \in X - \text{gspCl}(\{x\})$ , where  $X - \text{gspCl}(\{x\})$  is  $\text{gsp-open}$  set which does not contain  $x$ . This shows that  $X$  is  $\text{gsp-T}_0$ .

Next, we give the following

**Theorem 3.6:** A space  $X$  is  $\text{gsp-T}_0$  iff  $\text{gspCl}(\{x\}) \neq \text{gspCl}(\{y\})$  for every pair of distinct points  $x, y$  of  $X$ .

**Proof** follows from Th.3.5.

**Theorem 3.7 :** Every sub space of an  $\text{gsp-T}_0$  space is  $\text{gsp-T}_0$  space.

**Proof:** Let  $X$  be a space and  $(Y, \tau^*)$  be a subspace of  $X$  where  $\tau^*$  is the relative topology of  $\tau$  on  $Y$ . Let  $x, y$  be two distinct points of  $Y$ . As  $Y \subset X$ ,  $x$  and  $y$  are distinct points  $X$ . Since  $X$  is an  $\text{gsp-T}_0$  space, there exists an  $\text{gsp-open}$  set  $G$  such that  $x \in G$  but  $y \notin G$ . Then  $G \cap Y$  is an  $\text{gsp-open}$  set in  $(Y, \tau^*)$  which contains  $x$  but does not contain  $y$ . Hence  $(Y, \tau^*)$  is an  $\text{gsp-T}_0$  space.

We, define the following

**Definition 3.8:** A function  $f: X \rightarrow Y$  is said to be point -gspclosure 1-1 iff  $x, y \in X$  such that  $\text{gspCl}(\{x\}) \neq \text{gspCl}(\{y\})$  then  $f(\text{gspCl}(\{x\})) \neq f(\text{gspCl}(\{y\}))$

**Theorem 3.9:** If function  $f: X \rightarrow Y$  is point -gspclosure 1-1 and  $X$  is  $\text{gsp-T}_0$  then  $f$  is 1-1

**Proof:** Let  $x, y \in X$  with  $x \neq y$ . Since  $X$  is  $\text{gsp-T}_0$ , then  $\text{gspCl}(\{x\}) \neq \text{gspCl}(\{y\})$  by Theorem 3.6. But  $f$  is point -gspclosure 1-1 implies that  $f(\text{gspCl}(\{x\})) \neq f(\text{gspCl}(\{y\}))$ . Hence  $f(x) \neq f(y)$ . Thus,  $f$  is 1-1.

**Theorem 3.10:** Let  $f: X \rightarrow Y$  be a mapping from  $\text{gsp-T}_0$  space  $X$  into  $\text{gsp-T}_0$  space  $Y$ . Then  $f$  is point-gspclosure 1-1 iff  $f$  is 1-1

**Prof** follows from Theorem 3.5 above

**Theorem 3.11:** Let  $f: X \rightarrow Y$  be an injective  $\text{gsp-irresolute}$  mapping. If  $Y$  is  $\text{gsp-T}_0$  then  $X$  is  $\text{gsp-T}_0$ .

**Proof:** Let  $x, y \in X$  with  $x \neq y$ . Since  $f$  is injective and  $Y$  is  $\text{gsp-T}_0$ , there exists a  $\text{gspopen}$  set  $V_x$  in  $Y$  such that  $f(x) \in V_x$  and  $f(y) \notin V_x$  or there exists a  $\text{gspopen}$  set  $V_y$  in  $Y$  such that  $f(y) \in V_y$  and  $f(x) \notin V_y$  with  $f(x) \neq f(y)$ . By  $\text{gsp-irresoluteness}$  of  $f$ ,  $f^{-1}(V_x)$  is  $\text{gspopen}$  set in  $X$  such that  $x \in f^{-1}(V_x)$  and  $y \notin f^{-1}(V_x)$  or  $f^{-1}(V_y)$  is  $\text{gspopen}$  set in  $X$  such that  $y \in f^{-1}(V_y)$  and  $x \notin f^{-1}(V_y)$ . This shows that  $X$  is  $\text{gsp-T}_0$ .

We define the following mapping analogous to always semi-pre-open mapping.

**Definition 3.12 :** A mapping  $f: X \rightarrow Y$  is said to be always  $\text{gsp-open}$ , if the image of every  $\text{gsp-open}$  set of  $X$  is  $\text{gsp-open}$  in  $Y$ .

**Lemma 3.13 :** The property of a space being  $\text{gsp-T}_0$  is preserved under one-one, onto and always  $\text{gsp-open}$  mapping.

**Proof:** Let  $X$  be a  $\text{gsp-T}_0$  space and  $Y$  be any topological space. Let  $f: X \rightarrow Y$  be a one-one, onto always  $\text{gsp-open}$  mapping from  $X$  to  $Y$ . Let  $u, v \in Y$  with  $u \neq v$ . Since  $f$  is one-one, onto, there exist distinct points  $x, y \in X$ . Such that  $f(x) = u$ ,  $f(y) = v$ . Since  $X$  is on  $\text{gsp-T}_0$  space. There exists  $\text{gsp-open}$  set  $G$  in  $X$  such that  $x \in G$  but  $y \notin G$ . Since  $f$  is always  $\text{gsp-open}$ ,  $f(G)$  is an  $\text{gsp-open}$  set containing  $f(x) = u$  but not containing  $f(y) = v$ . Thus there exists an  $\text{gsp-open}$  set  $f(G)$  in  $y$  such that  $u \in f(G)$  but  $v \notin f(G)$  and hence  $Y$  is an  $\text{gsp-T}_0$  space.

Next, we define the following.

**Definition 3.14 :** A sub set  $A$  of a space  $X$  is called a  $\text{gspD-set}$  if there are two  $\text{gsp-open}$  subsets  $U$  and  $V$  such that  $U \neq X$  and  $A = U - X$ .

Clearly, every  $\text{gsp-open}$  set is  $\text{gspD-set}$ .

We, define the following

**Definition 3.15:** A space  $X$  is called a  $\text{gsp-D}_0$  if for any disjoint pair of points  $x$  and  $y$  of  $X$  there exists a  $\text{gspD-set}$  of  $X$  containing  $x$  but not  $y$  or a  $\text{gspD-set}$  of  $X$  containing  $y$  but not  $x$ .

Clearly, every  $\text{gsp-T}_0$  space is  $\text{gsp-D}_0$  space.

We prove the following

**Theorem 3.16 :** If  $f: X \rightarrow Y$  is  $\text{gsp-irresolute}$  surjective function and  $A$  is a  $\text{gspD-set}$  in  $Y$ , then the inverse image of  $A$  is a  $\text{gspD-set}$  in  $X$ .

**Proof:** Let  $A$  be a gspD-set in  $Y$ . Then there are gsp-open sets  $U_1$  and  $U_2$  in  $Y$  such that  $A = U_1 - U_2$  and  $U_1 \neq Y$ . By the gsp-irresoluteness of  $f$ ,  $f^{-1}(U_1)$  and  $f^{-1}(U_2)$  are gsp-open set in  $X$ . Since  $U_1 \neq Y$ , we have  $f^{-1}(U_1) \neq X$ . Hence  $f^{-1}(A) = f^{-1}(U_1) - f^{-1}(U_2)$  is a gspD-set.

We define the following .

**Definition 3.17 :** A space  $(X, \tau)$  is gsp- $T_1$  if and only if for  $x, y \in X$  such that  $x \neq y$ , there exists a gsp-open set containing  $x$  but not  $y$  and there is a gsp-open set containing  $y$  but not  $x$ .

It is easy to verify the following :

- (i) Every semipre- $T_1$  space is an gsp- $T_1$  space
- (ii) Every gsp- $T_1$  space is an gsp- $T_0$  space
- (iii) Every sg- $T_1$  space is an gsp- $T_1$  space . Also,

In view of above Note-3.3 , we have the following implication :

$$\text{ag-}T_1\text{-space} \rightarrow \text{gp-}T_1\text{-space} \rightarrow \text{gsp-}T_1\text{-space}$$

**Theorem 3.18 :** A space  $X$  is an gsp- $T_1$  space if and only if  $\{x\}$  is gsp-closed in  $X$  for every  $x \in X$ .

**Proof:** Let  $x, y$  be two distinct points  $X$  such that  $\{x\}$  and  $\{y\}$  are gsp-closed. Then  $X - \{x\}$  and  $Y - \{y\}$  are gsp-open in  $X$  such that  $y \in X - \{x\}$  but  $x \notin X - \{x\}$  and  $x \in X - \{y\}$  but  $y \notin X - \{y\}$ . Hence,  $X$  is an gsp- $T_1$  space.

Conversely, let  $X$  be an gsp- $T_1$  space and  $x$  be any arbitrary point of  $X$ . If  $y \in X - \{x\}$ , then  $y \neq x$ . Now the space being gsp- $T_1$  and  $y$  is a point different from  $x$ , there exists an gsp-open set  $G_y$  such that  $y \in G_y$  but  $x \notin G_y$ . Thus for each  $y \in X - \{x\}$ , there exists an gsp-open set  $G_y$  such that  $y \in G_y \subset X - \{x\}$ . Therefore  $\bigcup \{G_y \mid y \neq x\} \subset X - \{x\}$  which implies that

$$X - \{x\} \subset \bigcup \{G_y \mid y \neq x\} \subset X - \{x\}.$$

Therefore ,  $X - \{x\} = \bigcup \{G_y \mid y \neq x\}$ . Since  $G_y$  gsp-open in  $X$  and the union of gsp-open sets in  $X$  is gsp-open in  $X$ ,  $X - \{x\}$  is gsp-open in  $X$  and so  $\{x\}$  is gsp-closed.

Recall the following.

**Definition 3.19 [11] :** A topological space  $(X, \tau)$  is ags-symmetric if for any  $x$  and  $y$  in  $X$ ,  $x \in \text{agsCl}(\{y\})$  implies  $y \in \text{agsCl}(\{x\})$ .

We , define the following .

**Definition 3.20 :** A topological space  $(X, \tau)$  is semipre symmetric if for  $x$  and  $y$  in  $x$ ,  $x \in \text{semipreCl}(\{y\})$  implies  $y \in \text{semipreCl}(\{x\})$ .

**Definition 3.21 :** A topological space  $(X, \tau)$  is gsp-symmetric if for any  $x$  and  $y$  in  $X$ ,  $x \in \text{gspCl}(\{y\})$  implies  $y \in \text{gspCl}(\{x\})$ .

Clearly , every semipre-symmetric space is gsp-symmetric space.

**Theorem 3.22 :** If  $\{x\}$  is gsp-closed for each  $x$  in  $X$  then a space  $X$  is semipre-symmetric.

**Proof:** Suppose  $x \in \text{gspCl}(\{y\})$  and  $y \notin \text{gspCl}(\{x\})$ . Since  $\{y\} \subset X - \text{gspCl}(\{x\})$  and  $\{y\}$  is gsp-closed,  $\text{gspCl}(\{y\}) \subset X - \text{gspCl}(\{x\})$ . Thus  $x \in X - \text{gspCl}(\{y\})$ , a contradiction.

**Theorem 3.23 :** If a space  $X$  is extremely disconnected (i.e., closure of every open set is open) and semipre-symmetric, then  $\{x\}$  is gsp-closed, for each  $x$  in  $X$ .

**Proof:** Suppose  $\{x\} \subset U$  where  $U$  is semipre-open and  $\text{gspCl}(\{x\}) \not\subset U$ .

Then  $\text{gspCl}(\{x\}) \cap (X - U) \neq \emptyset$  Let  $y \in \text{gspCl}(\{x\}) \cap (X - U)$ . We have  $x \in \text{gspCl}(\{x\}) \subset (X - U)$  and  $x \notin U$ , a contradiction. Hence  $\{x\}$  is gsp-closed in  $X$ .

**Corollary 3.24 :** If  $X$  is extremely disconnected, then  $X$  is gsp- $T_1$  if and only if  $X$  is semipre-symmetric.

**Proof:** Obvious

Next, we have the following invariant properties.

**Theorem 3.25 :** Let  $f: X \rightarrow Y$  be an gsp-irresolutes injective map. If  $Y$  is gsp- $T_1$ , then  $X$  is gsp- $T_1$ .

**Proof:** Assume that  $Y$  is gsp- $T_1$ . Let  $x, y \in Y$  be such that  $x \neq y$ . Then there exists a pair of gsp-open sets  $u, v$  in  $Y$  such that  $f(x) \in u$ ,  $f(y) \in v$  and  $f(x) \notin v$ ,  $f(y) \notin u$ . Then  $x \in f^{-1}(u)$ ,  $y \in f^{-1}(v)$ ,  $x \notin f^{-1}(v)$  and  $y \notin f^{-1}(u)$ . Since  $f$  is gsp-irresolute,  $X$  is gsp- $T_1$ .

**Corollary 3.26:** A topological space  $(X, \tau)$  is  $\text{gsp-T}_1$  if and only if every finite subset of  $X$  is  $\text{gsp-closed}$ .

We, define the following

**Definition 3.27 :** A space  $X$  is called  $\text{gsp-D}_1$  if for any distinct pair of points  $x$  and  $y$  of  $X$  there exists a  $\text{gsp-D}$  set of  $X$  containing  $x$  but  $y$  and a  $\text{gsp-D}$  set of  $X$  containing  $y$  but not  $x$ .

Clearly, every  $\text{gsp-T}_1$  space is  $\text{gsp-D}_1$  space.

**Theorem 3.28:** If  $Y$  is a  $\text{gsp-D}_1$  and  $f: X \rightarrow Y$  is  $\text{gsp-irresolute}$  and bijective, then  $X$  is  $\text{gsp-D}_1$ .

**Proof:** Suppose that  $Y$  is a  $\text{gsp-D}_1$  space. Let  $x$  and  $y$  be any pair of distinct points in  $X$ . Since  $f$  is injective and  $Y$  is  $\text{gsp-D}_1$ , there exist  $\text{gsp-D}$  sets  $G_x$  and  $G_y$  of  $Y$  containing  $f(x)$  and  $f(y)$  respectively, such that  $f(y) \notin G_x$  and  $f(x) \notin G_y$ . By Theorem 3.16,  $f^{-1}(G_x)$  and  $f^{-1}(G_y)$  are  $\text{gsp-D}$  sets in  $X$  containing  $x$  and  $y$  respectively. This implies that  $X$  is a  $\text{gsp-D}_1$  space.

We, define and study the concept of  $\text{gsp-R}_0$  spaces in the following :

**Definition 3.29:** Let  $X$  be a topological space and  $A \subset X$ . Then the generalized pre-kernel of  $A$  denoted by  $\text{gsp-ker}(A)$ , is defined to be the set  $\text{gsp-ker}(A) = \{G \in \text{GSPO}(X) | A \subset G\}$

**Lemma 3.30 :** Let  $X$  be a topological space and  $x \in X$ . Then  $y \in \text{gsp-ker}(\{x\})$  if and only if  $x \in \text{gspCl}(\{y\})$

**Proof:** Suppose that  $y \in \text{gsp-ker}(\{x\})$ . Then there exists a  $\text{gsp-open}$  set  $V$  containing  $x$  such that  $y \notin V$ . Therefore, we have  $x \notin \text{gspCl}(\{y\})$ .

Conversely, Suppose that  $x \notin \text{gsp-ker}(\{y\})$ . Then there exists a  $\text{gsp-open}$  set  $V$  containing  $y$  such that  $x \notin V$ . Therefore, we have  $y \notin \text{gspCl}(\{x\})$ .

**Lemma.3.31:** Let  $X$  be a topological space and  $A$  be a subset of  $X$ . Then  $\text{gsp-ker}(A) = \{x \in X | \text{gspCl}(\{x\}) \cap A \neq \emptyset\}$

**Proof:** Let  $x \in \text{gsp-ker}(A)$  and suppose  $\text{gspCl}(\{x\}) \cap A = \emptyset$ . Hence  $x \notin \text{gspCl}(\{x\})$  which is a  $\text{gsp-open}$  set containing  $A$ . This is absurd. Since  $x \in \text{gsp-ker}(A)$ , consequently,  $\text{gspCl}(\{x\}) \cap A \neq \emptyset$ . Next, let  $\text{gspCl}(\{x\}) \cap A \neq \emptyset$  and suppose that  $x \notin \text{gsp-ker}(A)$ . Then there exists  $\text{gsp-open}$  set  $U$  containing  $A$  and  $x \notin U$ . Let  $y \in \text{gspCl}(\{x\}) \cap A$ , hence,  $U$  is a  $\text{gsp-neighbourhood}$  of  $y$  where  $x \notin U$ . But this is a contradiction, Therefore  $x \in \text{gsp-ker}(A)$  and the claim.

Now, we define the following

**Definition 3.32:** A space  $X$  is said to be  $\text{gsp-R}_0$  space if every  $\text{gsp-open}$  set contains the  $\text{gsp-closure}$  of each of its singletons.

Clearly, every  $\text{gsp-R}_0$  space is  $\text{gsp-T}_1$  space.

We recall the following :

**Definition 3.33 [12]:** A topological space  $(X, \tau)$  is said to be  $\text{gp-R}_0$  space if every  $\text{gp-open}$  set contains the  $\text{gp-closure}$  of each of its singletons.

**Definition 3.34 [3]:** A topological space  $X$  is said to be  $\text{ag-R}_0$  space if  $\text{agCl}(\{x\}) \subset U$  Whenever  $U$  is  $\text{ag-open}$  and  $x \in U$ .

Hence, we have the following w.r.t. Note-3.3:

$$\text{ag-R}_0\text{-space} \rightarrow \text{gp-R}_0\text{-space} \rightarrow \text{gsp-R}_0\text{-space}$$

Now, we characterize the  $\text{gsp-R}_0$  spaces in the following.

**Theorem 3.35:** For any topological space  $X$  the following properties are equivalent:

- (i)  $X$  is  $\text{gsp-R}_0$  space;
- (ii) For any  $F \in \text{GSPC}(X, \tau)$   $x \notin F \Rightarrow F \subset U$  and  $x \notin U$  for some  $U \in \text{GSPC}(X, \tau)$ ;
- (iii) For any  $F \in \text{GSPC}(X, \tau)$   $x \notin F \Rightarrow F \cap \text{gspCl}(\{x\}) = \emptyset$ ;
- (iv) For any distinct points  $x$  and  $y$  either  $\text{gspCl}(\{x\}) = \text{gspCl}(\{y\})$  or  $\text{gspCl}(\{x\}) \cap \text{gspCl}(\{y\}) = \emptyset$ .

**Proof: (i)  $\Rightarrow$  (ii):** Suppose  $F \in \text{GSPC}(X, \tau)$  and  $x \notin F$ . Then by (i)  $\text{gspCl}(\{x\}) \subset X \setminus F$ . Set  $U = X \setminus \text{gspCl}(\{x\})$  then  $U \in \text{GSPC}(X, \tau)$ ,  $F \subset U$  and  $x \notin U$

**(ii)  $\Rightarrow$  (iii):** Let  $F \in \text{GSPC}(X, \tau)$ ,  $x \notin F$ . Therefore, there exists  $U \in \text{GSPC}(X, \tau)$  such that  $F \subset U$  and  $x \notin U$ . Since  $U \in \text{GSPC}(X, \tau)$ ,  $U \cap \text{gspCl}(\{x\}) = \emptyset$ . and  $F \cap \text{gspCl}(\{x\}) = \emptyset$ .

**(iii)  $\Rightarrow$  (iv):** Suppose that  $\text{gspCl}(\{x\}) \neq \text{gspCl}(\{y\})$  for distinct points  $x, y \in X$ . There exist  $z \in \text{gspCl}(\{x\})$  such that  $z \notin \text{gspCl}(\{y\})$ . One can also assume that  $z \in \text{gspCl}(\{y\})$  such that  $z \notin \text{gspCl}(\{x\})$ . There exists

$V \in \text{GSPPO}(X, \mathfrak{T})$  such that  $y \notin V$  and  $z \in V$ . Hence  $x \in V$ . Therefore we obtain  $x \notin \text{gspCl}(\{y\})$ . By (iii) we obtain  $\text{gspCl}(\{x\}) \cap \text{gspCl}(\{y\}) = \emptyset$ . The proof of otherwise is similar.

(iv)  $\Rightarrow$  (i): Let  $V \in \text{GSPPO}(X, \mathfrak{T})$  and  $x \in V$ . For each  $y \notin V$ ,  $x \neq y$  and  $x \notin \text{gspCl}(\{y\})$ . This show that  $\text{gspCl}(\{x\}) \neq \text{gspCl}(\{y\})$ . By (iv)  $\text{gspCl}(\{x\}) \cap \text{gspCl}(\{y\}) = \emptyset$  for each  $y \in X|V$ . Hence  $\text{gspCl}(\{x\}) \cap (\cup\{\text{gspCl}(\{y\}) \mid y \in X|V\}) = \emptyset$ . On the other hand, since  $V \in \text{GSPPO}(X, \mathfrak{T})$  and  $y \notin X|V$ . We have  $\text{gspCl}(\{y\}) \subset X|V$ . Therefore  $X|V = \cup\{\text{gspCl}(\{y\}) \mid y \in X|V\}$ . Therefore we obtain  $(X|V) \cap \text{gspCl}(\{x\}) = \emptyset$  and  $\text{gspCl}(\{x\}) \subseteq V$ . Hence  $(X, \mathfrak{T})$  is  $\text{gsp-R}_0$  space.

Finally, we define and study the following.

**Definition 3.36:** A space  $X$  is said to be a  $\text{gsp-R}_1$  if for  $x, y$  in  $X$  with  $\text{gspCl}(\{x\}) \neq \text{gspCl}(\{y\})$ , there exists disjoint  $\text{gsp}$ -open sets  $U$  and  $V$  such that  $\text{gspCl}(\{x\}) \subset U$  and  $\text{gspCl}(\{y\}) \subset V$ .

We recall the following

**Definition 3.37[12] :** A topological space  $X$  is said to be  $\text{gp-R}_1$  space if for  $x, y$  in  $X$  with  $\text{gpCl}(\{x\}) \neq \text{gpCl}(\{y\})$ , there exist disjoint  $\text{gp}$ -open sets  $U$  and  $V$  such that  $\text{gpCl}(\{x\})$  is a subset of  $U$  and  $\text{gpCl}(\{y\})$  is a subset of  $V$ .

**Definition 3.38[3] :** A topological space  $X$  is said to be  $\alpha\text{g-R}_1$  space if for  $x, y$  in  $X$  with  $\alpha\text{gCl}(\{x\}) \neq \alpha\text{gCl}(\{y\})$ , there exist disjoint  $\alpha\text{g}$ -open sets  $U$  and  $V$  such that  $\alpha\text{gCl}(\{x\})$  is a subset of  $U$  and  $\alpha\text{gCl}(\{y\})$  is a subset of  $V$

In view of Note-3.3, we have the following :

$\alpha\text{g-R}_1\text{-space} \rightarrow \text{gp-R}_1\text{-space} \rightarrow \text{gsp-R}_1\text{-space}$

We, prove the following

**Theorem 3.39:** If  $X$  is  $\text{gsp-R}_1$ , then  $X$  is  $\text{gsp-R}_0$ -space.

Proof: Let  $U$  be a  $\text{gsp}$ -open and  $x \in U$ . If  $y \notin U$  then since  $x \notin \text{gspCl}(\{y\})$ ,  $\text{gspCl}(\{x\}) \cap \text{gspCl}(\{y\}) = \emptyset$ . Hence there exists a  $\text{gsp}$ -open  $V$  such that  $\text{gspCl}(\{x\}) \subset V$  and  $x \notin V$ , which implies  $y \notin \text{gspCl}(\{x\})$ . Thus  $\text{gspCl}(\{y\}) \subset U$ . Therefore  $(X, \mathfrak{T})$  is  $\text{gsp-R}_0$  space.

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