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Note on Nice Elongations of QTAG-Modules

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ABSTRACT*: A right module M over an associative ring with unity is a QTAG-module if every finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules. In this paper we examine the relationship between the classes of n-layered modules and strong ω-elongations of summable modules by (ω + k)-projective modules and whether there is an analogue with the strong ω–elongations of a totally projective modules by (ω + k)-projective modules.*
KEYWORDS*: QTAG Modules, n-layered modules, $(\omega+k)$-projective modules, summable modules and $\sum$-modules.*

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I. INTRODUCTION

Many concepts for groups like purity, projectivity, injectivity, height etc. have been generalized for modules.Toobtain results of groups which are not true for modules either conditions have been applied on modules or upon theunderlying rings. We imposed the condition on modules thatevery finitely generated submodule of any homomorphic image of the module is a direct sum of uniserial moduleswhile the rings are associative with unity. After these conditions many elegant results of groups can be proved forQTAG-modules which are not true in general. Many results of this paper are motivated from the paper [1].

The study of QTAG-modules was initiated by Singh [2]. Mehdi, Abbasi etc. worked a lot on this module. They studied different notions and structures on QTAG-modules and developed the theory of these modules by introducing several notions and some interesting properties of these modules and characterized different submodules of QTAG-modules.Yet there is much to explore.

Throughout this paper, all rings will be associative with unity and modules M areunital QTAG-modules. An element $x \in M$ is uniform, if xR is a non-zero uniform(hence uniserial) module and for any R-module M with a unique composition series,d(M) denotes its composition length. For a uniform element $x \in M$; $e(x) = d(xR)$and $H_M(x) = \sup\left\{d\left(\frac{yR}{xR}\right) \middle| y \in M, x \in yR \text{ and } y \text{ uniform}\right\}$are the exponent and heightof x in M; respectively. $H_k(M)$denotes the submodule of M generated by the elements ofheight at least k and $H^k(M)$ is the submodule of M generated by the elements of exponents at most. A QTAG-module M is called a $\sum$-module, if all of its high submodules are direct sums of uniserial modules. M is h-divisible if $M = M^1 = \bigcap_{k=0}^{\infty} H_k(M)$.

A module M is summable if $Soc(M) = \bigoplus_{\alpha<\tau} S_\alpha$,where S_α is the set of all elements of $H_\alpha(M)$ which are not in $H_{\alpha+1}(M)$, where τ is the length of M.

A submodule N⊂M is nice [3, Definition 2.3] in M, if $H_\sigma(M/N) = (H_\sigma(M) + N)/N$for all ordinals σ, i.e. every coset of M modulo N may be represented by an element of the same height.

Recall that a QTAG-module M is (ω + 1)-projective if there exists a submodule $N \subset H^k(M)$such that M/Nisadirect sum of uniserial modules and aQTAG module M is (ω + k)-projective if there exists a submodule $N \subset H^k(M)$ such that $M/N$is a direct sum of uniserial modules[4]. .Notations and terminology are followed from [5,6].

A QTAG-module is an ω-elongation of a totally projective QTAG-module by a $(\omega+k)$ −projective QTAG-module if and only if $H_w(M)$ is totally projective and $M/H_\omega(M)$ is $(\omega+k)$-projective.

II. MAIN RESULTS

In [7], we treat a more general situation by studying the strong ω-elongations of summable modules by $(\omega+k)$-projectiveQTAG-modules.Specifically the moduleMis such a specialω-elongation if M^1 is summable and there exists a submodule $N \subseteq H^k(M)$such that $M/(N + M^1)$is a direct sum of uniserial modules. We showed there that under certain additional circumstances on Nthese elongations are of necessity summable modules; in fact, $N \cap H_k(M) \subseteq M^1$was taken, that is Nis a height finite submodule of M.

In [8], the class of n-layered module was introduced which is a proper subclass of Σ-modules as follows: Mis an n-layered module if $H^k(M) = \bigcup_{i<\omega} M_i$, $M_i \subseteq M_{i+1} \subseteq H^k(M), \forall\, i \geq 1$. $M_i \cap H_i(M) \subseteq M^1$. We also proved there that every n-layered module which is a strong ω-elongation of a totally projective module by a (ω+k)-projective module is totally projective and vice-versa;in particular each n-layered module is (ω+k)-projective uniquely when it is a direct sum of modules of length at most (ω) +n.

The aim of the present paper is to examine what is the relationship between the classes of n-layered modules and strong ω-elongations of summable modules by (ω+k)-projective modules i.e., how n-layered modules are situated inside these special ω-elongations of summable modules by (ω+k)-projective modules and whether there is ananalogue with the strong ω-elongations of a totally projective modules by(ω+k)-projective modules.

Before doing that, we need the following preliminaries:

A module M is said to be pillared [9], provided that M/M^1 is a direct sum of uniserial modules. Clearly, such a module is necessarily n-layered module and hence a Σ-module [8], whereas the converse implications failed.The next theorems answers under what conditions converse holds true.

A module M is said to be a strong ω-elongation (of a summable module) by a (ω+k)- projective module if there exists a submodule $N \subseteq H^k(M)$ with $M/(N+M^1)$is a direct sum of uniserial modules. Such a module has first Ulm-factor which is of necessity (ω + k)-projective, while this property is not retained in a converse way that is there is a module (ω + k)-projective first Ulm-factor which is not a strong ω-elongation by a (ω+k)-projective module. That is why we have also named these modules as with strong (ω + k)-projective firstUlm-factor.

Now we are in the state to prove the following main result:

Theorem2.1. An n-layered module is a strong ω-elongation by a (ω+k)-projective module if and only if it is a pillared module.

Proof. Suppose that M is a n-layered module, so $H^k(M) = \bigcup_{i<\omega} M_i$, $M_i \subseteq M_{i+1} \subseteq H^k(M)\ \forall\, i \geq 1$. $M_i \cap H_i(M) \subseteq M^1$and suppose there exists a submoduleN of M such that $N \subseteq H^k(M)$ with $M/(N+M^1)$is a direct sum of uniserial modules. Clearly,

$$\frac{M/M^1}{(N+M^1)/M^1} \cong \frac{M}{N+M^1}$$

SinceN $\subseteq \bigcup_{i<\omega} M_i$, we infer that $N = \bigcup_{i<\omega}(M_i \cap N)$ and thus $\frac{(N+M^1)}{M^1} = \cup\,(\frac{(N+M^1)}{M^1})$ by putting $N_i = M_i \cap N$.

Using modular law, we compute that

$$\frac{N_i + M^1}{M^1} \cap H_i\left(\frac{M}{M^1}\right) = \frac{\left(N_i \cap H_i(M)\right) + M^1}{M^1}$$
$$= \{0\}$$

Hence $M/M^1$is a direct sum of uniserial modules and therefore M is pillared. The proof is complete as the converse is trivial. □

Proposition 2.1. An n-layered module is a strong ω-elongation of a summable module by a (ω+k)-projective module if and only if it is a summable pillared module.

Proof. Suppose n-layered module is a strong ω-elongation of a summable module by a (ω + k)-projective module. Since M is a Σ-module and $M^1$is summable so M has to be summable as well and Theorem 2.1 ensures that M^1 must be pillared.

Suppose that M is summable pillared module, so it ensures that M^1 is summable and pillared modules are both n-layered and strong ω-elongations by(ω+k)-projective module by taking N = 0, which complete the proof. □

As an immediate consequence of the above, we have the following corollary:

Corollary2.1. Suppose M is a Σ-module which is a strong ω-elongation of a summable module by a (ω + k)-projective module and the (ω + n)-th Ulm-Kaplansky invariants of M are zero for each n such that 0 ≤ n ≤ k − 1, if k > 1. Then M is a summable pillared module.

Proof. Since $(\omega+n)$-th Ulm-Kaplansky invariants of Mare zero so, $H^k(M) = H^k(N) \oplus H^k(M^1)$whereNis a high submodule of M. Since it is a direct sum uniserial module, $H^k(N) = \bigcup_{i<\omega} N_i$, $N_i \subseteq N_{i+1} \subseteq H^k(N)$ where $N_i \cap H_i(N) = 0$. Putting $M_i = N_i \oplus H^k(M_1)$ we obtain $H^K(M) = \bigcup_{i<\omega} M_i$. Since N is h-pure in M, we have

$$\begin{aligned} M_i \cap H_i(M) &\subseteq M^1 + N_i \cap H_i(M) \\ &= M^1 + N_i \cap H_i(M) \\ &= M^1 \end{aligned}$$

By Proposition 2.1, M is an n-layered module. □

Proposition2.2. A module of length not exceeding (ω + n − 1) is an n-layered module if and only if it is a direct sum of countable modules..

Proof. Suppose Mbe an n-layered module of length not exceeding $(\omega+n-1)$, then$M^1 \subseteq H^{n-1}(M)$ and hence

$$Soc\left(\frac{M}{M^1}\right) = \bigcap_{i<\omega} \frac{H_i(M) + Soc(M)}{M^1} \subseteq \frac{H^n(M)}{M^1}$$

Since $H(\bigcap_{i<\omega}(H_i(M + Soc(M))) \subseteq M^1$. Moreover, we write $H^n(M) = \bigcup_{i<\omega} M_i$, $M_i \subseteq M_{i+1} \subseteq H^n(M)$ and $M_i \cap H_i(M) \subseteq M^1$. Consequently,

$Soc\left(\frac{M}{M^1}\right) = \bigcup_{i<\omega} T_i$, where $T_i = \frac{M_i + M^1}{M^1} \cap Soc\left(\frac{M}{M^1}\right)$. We compute

$$\begin{aligned} T_i \cap H_i(M/M^1) &= T_i \cap \left(\frac{H_i(M)}{M^1}\right) \\ &= \frac{(M_i + M^1) \cap H_i(M)}{M^1} \\ &= \frac{\left(M_i \cap H_i(M)\right) + M^1}{M^1} \\ &= 0 \end{aligned}$$

hence M is pillared and as $M^1$is bounded, Misadirect sum of countable modules. The sufficiency is obvious and follows from [8].

As an immediate consequence of the above, we have the following corollary: □

Corollary2.2. A module is an n-layered module if and only if its each (ω + n − 1)-high submodules is a direct sum of countable modules.

Proof. Let Mbe an n-layered module and N be its (ω + n − 1)-high submodule. In [8], we show that M is an n-layered module precisely when N is an n-layered module. Hence using the Proposition 2.2, result follows immediately. □

With the help of the last statement we can verify once again the validity of Corollary 2.1 as it is checked that a submodule N of M is h-high in M if and only if N is (ω+n−1)- high in M whenever (ω+m)-th Ulm Kaplansky invariants of M are zero for0≤m≤ n − 1, that is

$$Soc\big(H_\omega(M)\big) = \cdots = Soc\big(H_{\omega+n-1}(M)\big).$$

A module M is said to be strong (ω + n −1)-elongation of a summable module by a totally projective module if $H_{\omega+n-1}(M)$ is summable and there is a nice submodule Kof M such that K ∩ $H_{\omega+n-1}(M)$ = 0 and $M/(K \oplus H_{\omega+n-1}(M))$ is totally projective.[4]

Now we are in the state to prove our final result.

Theorem2.2 An n-layered module is a strong (ω + n − 1)-elongation of a summable module by a totally projective module if and only if it is a summable pillared module.

Proof. Clearly

$$\frac{M}{K \oplus H_{\omega+n-1}(M)} \cong \frac{M/H_{\omega+n-1}(M)}{(K \oplus H_{\omega+n-1}(M))/H_{\omega+n-1}(M)}$$

is totally projective. Since $K \cap H_{\omega+n-1}(M) = 0$, K is contained in some $(\omega + n - 1)$-high submodule of M, say N. By Corollary 2.2, N is totally projective of length at most $(\omega + n - 1)$, so we may write $N + \bigcup_{i<\omega} N_i$, where $N_i \subseteq N_{i+1} \subseteq N$ and all N_i's are height finite in N and hence in Mas N is isotype in M. Therefore

$$\frac{K \oplus H_{\omega+n-1}(M)}{H_{\omega+n-1}(M)} = \bigcup_{i<\omega}\left[\left(\frac{N_i + H_{\omega+n-1}(M)}{H_{\omega+n-1}(M)}\right) \cap \left(\frac{K \oplus H_{\omega+n-1}(M)}{H_{\omega+n-1}(M)}\right)\right]$$

In the same way, we can show that $\frac{N_i + H_{\omega+n-1}(M)}{H_{\omega+n-1}(M)}$ are high finite in $\frac{M}{H_{\omega+n-1}(M)}$ and by [4]. $\frac{K \oplus H_{\omega+n-1}(M)}{H_{\omega+n-1}(M)}$ is nice in $\frac{M}{H_{\omega+n-1}(M)}$ and hence $\frac{M}{H_{\omega+n-1}(M)}$ is totally projective. Then

$$\frac{M}{H_\omega(M)} \cong \frac{M/H_{\omega+n-1}(M)}{H_\omega(M)/H_{\omega+n-1}(M)} = \frac{M/H_{\omega+n-1}(M)}{H_\omega(M/H_{\omega+n-1}(M))}$$

should be a direct sum of uniserial modules and henceMis pillared.

On the other hand, $H_{\omega+n-1}(M)$ being summable implies that $H_\omega(M)$is summable and hence M has to be summable, thus it is summable pillared, which completes the proof. □

As an immediate consequence for n = 1, we have the following:

Corollary2.3. A Σ-module is a strong ω-elongation of a summable module by a totally projective module if and only if it is a summable pillared module.

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