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The Use of Adomian Decomposition Method in Solving Second Order Autonomous and Non-autonomous Ordinary Differential Equations

Anthony Anya Okeke[1], Pius Tumba[2], Jeremiah Jerry Gambo[3]
[1, 2, 3]Department of Mathematics, Faculty of Science, Federal University Gashua,
P. M. B. 1005 Gashua, Yobe State, Nigeria
Corresponding Author: Anthony Anya Okeke

Abstract:*Many researchers have developed numerical techniques to solve initial value problems of ordinary differential equations since its discovery by Leonard Euler in 1768. Some have tried to improve on existing methods and their efficiencies such as stability, accuracy, convergence, and consistency. This paper examines the Adomian Decomposition Method for the solution of second-orderautonomous and non-autonomous ordinary differential equations. The results from the two numerical problems used shows that the Adomian Decomposition Method is almost the same as the theoretical solutions. These results obtained indicate that the method is efficient, reliable, and computationally stable.*

Keywords*: Ordinary Differential Equations (ODE), Numerical Methods, Adomian Decomposition Method (ADM), Initial Value Problems (IVP).*

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I. INTRODUCTION

It is a well-known fact that Differential Equations are among the most mathematical tools used in producing models in the engineering, mathematics, physics, aeronautical, elasticity, astronomy, dynamics, biology, chemistry, medicine, environmental sciences, econometrics, social sciences, banking and many other areas [1]. Many researchers have studied the nature of Differential Equations and many complicated systems that can be described quite precisely with mathematical expressions. It is very important to note that many differential equations cannot be solved algebraically (analytically) [2]. This means that the solution cannot be expressed as the sum of a finite number of elementary functions (polynomials, exponentials, trigonometric, and hyperbolic functions). Sometimes, it is possible to find an algebraic solution, but we may be faced with a system of thousands, even millions of differential equations. For simple differential equations, it is possible to find closed form solutions [3].

Historically, the ancestor of all numerical methods in use today was developed by Leonhard Euler between 1768 and 1770 [4]. The famous of Euler was republished in his collected works in 1913 [5]. Recalled the initial value problem

$y' = f(x, y), \qquad y(a) = \eta.$

Of all the computational methods for the numerical solution of this problem, the easiest to implement is Euler's rule [1],

$y_{n+1} - y_{n=hf(x_n,y_n)} \equiv hf_n, \quad n = 1,\ 2,\ \cdots,\ m$

where the step size $h = x_{n+1} - x_n$.

Since then, many authors have worked to improve on the Euler's methods because of its ease of implementation, others have advanced additional methods; among which are: Adams-Bashforth-Moulton methods, Nystrom type methods, the self-starting Runge-Kutta type method which involves several function evaluation per-step, Improved Euler method, the Linear multistep method, Taylor series method, Explicit Euler method, Hybrid method, Extrapolation method, Cyclic Composite Method, Methods for Stiff problems, Power Series Method and Block Procedure. From literature, theses numerical methods are suitable for solving some sets of initial value problems in ODEs.

The efficiency of any method in numerical analysis depends on the stability (zero-stability, weak-stability, absolute-stability), accuracy, convergence and consistency properties of the method. The accuracy properties of the different methods are usually compared by considering the order of convergence, truncation error coefficients, and computational simplicity as well as inexpensive of the method and effective for a wide range of ODEs.Among researchers that have developed numerical schemes for solving initial value problems are: [6, 7, 2, 8, 9, 10, 11].

The Adomian Decomposition Method is a semi-analytical method for solving linear or nonlinear and deterministic or stochastic operator equations, including ODEs, partial differential equations (PDEs), integral equations, integro-differential equations, etc. The method was developed from the 1970s to the 1990s by George Adomian, chair of the Center for Applied Mathematics at the University of Georgia. The method generates a solution in the form of a series whose terms are determined by a recursive relationship using the Adomian Polynomials. According to [12], researchers who have used the ADM have frequently enumerated on the many advantages it offers. The Adomian decomposition method yields an efficient numerical solution with high degree accuracy. It enables the accurate and efficient analytical solution of the nonlinear differential equation without the need to resort to linearization or perturbation approaches. The method consists of splitting the given equation into linear and non-linear parts, the highest order derivative operator contained in the linear operator is inverted on both sides, the known function is decomposed into a series whose components can be easily computed.

However, from the time it was first presented, the ADM has led to several modifications on the method by various researchers in an attempt to improve the accuracy or expand the application of the original method. See [13, 14, 15, 16, 17].We shall proceed to discuss the basic theory and concepts of ADM with some numerical examples.

II. THE ADOMIAN DECOMPOSITION METHOD

We begin by giving a brief outline of the method.
If we begin with the equation

$$F = g \qquad (1)$$

whereF represents the general nonlinear ordinary differential operator involving both linear and nonlinear terms. The linear term is decomposedinto$L + R$, where L is easily invertible and R is the remainder of the linear operator of the order less than L as L may be taken as the highest order derivative. Thus, the equation may be written

$$L_y + R_y + N_y = g(x) \qquad (2)$$

whereN_y represent the nonlinear terms solving for L_y by making it subject of the formula, we get

$$L_y = g(x) - R_y - N_y \qquad (3)$$

By solving (3) for L_y, since L is invertible, we can write

$$L^{-1}L_y = L^{-1}g(x) - L^{-1}R_y - L^{-1}N_y \qquad (4)$$

For the initial value problems, we conveniently define L^{-1} for $L = \frac{d^n}{dx^n}$ as the $n -$ fold definite integration from 0 tox (i.e. $L^{-1}L_y = y - y(x_0) - (x - x_0)y'(x_0)$). If L is a second order operator L is a two-fold integral and so by solving (4) we get

$$y = A + Bx + L^{-1}g(x) - L^{-1}Ry - L^{-1}Ny \qquad (5)$$

whereA and B are constants of integrations and can be found from the initial boundary conditions.
The Adomian method consists of approximating the solution of (2) as an infinite series

$$y(x) = \sum_{n=0}^{\infty} y_n(x) \qquad (6)$$

And decomposing the nonlinear operator Ny into a series

$$N(y) = \sum_{n=0}^{\infty} A_n \qquad (7)$$

Where A_n, depending on $y_0, y_1, y_2, \cdots y_n$, are called the Adomian polynomials, and are obtained for the nonlinear $N_y - g(y)$ by the definitional formula [Adomian 1983]

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{n} \lambda^i y_i\right)\right]_{\lambda=0}, n = 0, 1, 1, 2, \ldots \qquad (8)$$

Substituting (7) and (8) into (6), yields

$$\sum_{n=0}^{\infty} y_n = \varphi_0 + L^{-1}g(x) - L^{-1}R\left(\sum_{n=0}^{\infty} y_n\right) - L^{-1}\left(\sum_{n=0}^{\infty} A_n\right) \qquad (9)$$

where,

$$\varphi_0 = \begin{cases} y(0), & \text{if} L = \frac{d}{dx}, \\ y(0) + xy'(0), & \text{if} L = \frac{d^2}{dx^2}, \\ y(0) + xy'(0) + \frac{x^2}{2!}y''(0), & \text{if} L = \frac{d^3}{dx^3}, \\ \vdots \\ y(0) + x'(0) + \frac{x^2}{2!}y''(0) + \cdots \frac{x^n}{n!}y^n(0), & \text{if} L = \frac{d^{n+1}}{dx^{n+1}} \end{cases} \quad (10)$$

Therefore

$$\begin{cases} y_0 = \varphi_0 + L^{-1}g(x), \\ y_1 = -L^{-1}Ry_0 - L^{-1}A_0 \\ y_2 = -L^{-1}Ry_1 - L^{-1}A_1, \\ \vdots \\ y_{n+1} = -L^{-1}Ry_n - L^{-1}A_n, \ n \geq 0, \end{cases} \quad (11)$$

We write the first five Adomian polynomials

$$\begin{cases} A_0 = N(y_0), \\ A_1 = y_1 N'(y_0), \\ A_2 = y_2 N'(y_0) + \frac{1}{2!}y_1^2 N''(y_0), \\ A_3 = y_3 N'(y_0) + y_1 y_2 N''(y_0) + \frac{1}{3!}y_1^3 N'''(y_0), \\ A_4 = y_4 N'(y_0)\left[\frac{1}{2!}y_1^2 + y_1 y_3\right]N''(y_0) + \frac{1}{2!}y_1^2 y_2 N'''(y_0) + \frac{1}{4!}y_1^2 N''''(y_0) \\ \vdots \end{cases} \quad (12)$$

So, the practical solution for the n terms approximation is

$$y = \lim_{n=\infty} \Phi_n(y) \quad (13)$$

$$\Phi_n(y) = \sum_{i=0}^{n-1} y_i \quad (14)$$

III. NUMERICAL EXAMPLES AND RESULTS OF ADOMIAN DECOMPOSITION METHOD

In this section, we applied the ADM to solve two sample problems in ODEs and their numerical results are illustrated. In each problem, only the first ten terms of the decomposition series will be used in computing the results.

Problem 1:

We consider the second order differential equation of the form

$$y'' - y = 2, \quad y(0) = 0, \quad y'(0) = 1 \quad x\epsilon[0,6] \quad (15)$$

With the theoretical solution:

$$y(x) = \frac{3}{2}e^x + \frac{1}{2}e^{-x} - 2$$

We apply the ADM operator to equation (15) to produce

$$Ly'' = 2 + y \quad (16)$$

$$L = \frac{d^2y}{dx^2}$$

The inverse operator

$$L^{-1} = \int_0^x \int_0^x (.)\, dxdx$$

Applying L^{-1} to both side of (16) and impose the boundary conditions, we obtain

$$y(x) = y(0) + y'(0) + L^{-1}(2) + L^{-1}(y) \quad (18)$$

By using (9), we have

$$y(x) = x + 2\int_0^x \int_0^x dxdx + \int_0^x \int_0^x \sum_{n=0}^{\infty} y_n \quad (19)$$

The ADM introduces the recursive relation

$$y_0(x) = x + L^{-1}(2) + L^{-1}(y) = x + x^2$$

$$y_{n+1} = L^{-1}(y_n) n \geq 0$$

We can then proceed to compute the first few terms of the series:

$$y_1(x) = L^{-1}(y_0) = \int_0^x \int_0^x \sum_{n=0}^{\infty} (x + x^2)dxdx = \frac{x^3}{6} + \frac{x^4}{12}$$

$$y_2(x) = L^{-1}(y_1) = \int_0^x \int_0^x \left(\frac{x^3}{6} + \frac{x^4}{12}\right) dxdx = \frac{x^5}{120} + \frac{x^6}{360}$$

$$y_3(x) = L^{-1}(y_2) = \int_0^x \int_0^x \left(\frac{x^5}{120} + \frac{x^6}{360}\right) dxdx = \frac{x^7}{5040} + \frac{x^8}{20,160}$$

$$y_4(x) = L^{-1}(y_3) = \int_0^x \int_0^x \frac{x^7}{5040} + \frac{x^8}{20,160} dxdx = \frac{x^9}{362,880} + \frac{x^{10}}{1,814,400}$$

$$y_5(x) = L^{-1}(y_4) = \int_0^x \int_0^x \frac{x^9}{362,880} + \frac{x^{10}}{1,814,400} dxdx = \frac{x^{11}}{39,916,800} + \frac{x^{12}}{239,500,800}$$

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$$y_n(x) = \frac{x^{(2n+1)}}{(2n+1)!} + \frac{2\,.\,x^{(2n+2)}}{(2n+2)!} \qquad (20)$$

Hence,

$$\Phi_{10}(x) = \sum_{n=0}^{9} y_n(x) \qquad (21)$$

When $n = 10$ in this case, the results of the difference between the exact solution and the ADM solution along-side the Absolute error, E_A using a step size of 1 is as shown as in Table 1. Obviously, the results are in agreement with the exact solution and higher accuracy can be obtained by evaluating more components of the series (21).

Table I: Exact versus ADM solution of example 1 with step size of 0.5

X	Exact Solution	Solution with ADM	Absolut Error E_A
0.5	0.77634723591	0.77634723591	0.00000000000
1.0	2.26136246327	2.26136246327	0.00000000000
1.5	4.83409868558	4.83409868558	0.00000000000
2.0	9.15125179001	9.15125179001	0.00000000000
2.5	16.31478344037	16.31478344036	0.00000000001
3.0	28.15319891897	28.15319891870	0.00000000027
3.5	47.68827662975	47.68827662271	0.00000000704
4.0	79.90638286916	79.90638274805	0.00000012111
4.5	133.03125144905	133.03124995196	0.00000149709
5.0	220.62310762736	220.62309337112	0.00001425624
5.5	365.03994178205	365.03983185037	0.00010993168

With a step size of 0.1, we obtain a similar result as shown in table II

Table II: Exact versus ADM solution of example 1 with step size of 0.1

X	Exact Solution	Solution with ADM	Absolut Error E_A
0.1	0.11017508613	0.11017508613	0.00000000000
0.2	0.24146951378	0.24146951378	0.00000000000
0.3	0.39519732170	0.39519732170	0.00000000000
0.4	0.57289706948	0.57289706948	0.00000000000
0.5	0.77634723591	0.77634723591	0.00000000000
0.6	1.007584018633	1.00758401863	0.00000000000
0.7	1.26892171310	1.26892171310	0.00000000000
0.8	1.56297587480	1.56297587480	0.00000000000
0.9	1.89268949661	1.89268949661	0.00000000000
1.0	2.26136246327	2.26136246327	0.00000000000
1.1	2.67268457777	2.67268457777	0.00000000000

Problem 2:

We consider the second order differential equation of the form

$$y'' + y = 0, \quad y(0) = 2, \quad y'(0) = 3, \qquad x \in [0,4] \qquad (22)$$

The exact solution of (22) is $y(x) = 2\cos x + 3\sin x$. In an operator form, (22) becomes

$$Ly = -y \qquad (23)$$

$$y(x) = y(o) + xy'(0) - L^{-1}(y)$$

$$y(x) = 2 + 3x - L^{-1}(y)$$

$$\begin{cases} y_0(x) = 2 + 3x \\ y_{n+1} = -L^{-1}(y_n), \quad n \geq 0 \end{cases} \quad (24)$$

$$y_1(x) = -\int_0^x \int_0^x (2 + 3x)dxdx = -x^2 - \frac{x^3}{2}$$

$$y_2(x) = -\int_0^x \int_0^x \left(-x^2 - \frac{x^3}{2}\right) dxdx = \frac{x^4}{12} + \frac{x^5}{40}$$

$$y_3(x) = -\int_0^x \int_0^x \left(\frac{x^4}{12} + \frac{x^5}{40}\right) dxdx = -\frac{x^6}{360} - \frac{x^7}{1{,}680}$$

$$y_4(x) = -\int_0^x \int_0^x \left(-\frac{x^6}{360} - \frac{x^7}{1{,}680}\right) dxdx = \frac{x^8}{20{,}160} + \frac{x^9}{120{,}960}$$

$$y_5(x) = -\int_0^x \int_0^x \left(\frac{x^8}{20{,}160} + \frac{x^9}{120{,}960}\right) dxdx = -\frac{x^{10}}{1{,}814{,}400} - \frac{x^{11}}{13{,}305{,}600}$$

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$$y_n(x) = (-)^n \left(\frac{2x^{2n}}{(2n)!} + \frac{3x^{(2n+1)}}{(2n+1)!}\right)$$

Consequently,

$$\Phi_{15}(x) = \sum_{n=0}^{14} y_n(x) \quad (25)$$

Here, we use only the first fourteen terms in evaluating the approximate solution of equation (22). The exact solution and the ADM solution with various step sizes with the E_A is as given in table III and table IV respectively. However, the results of the ADM are almost the same as the exact solution.

Table III: Exact versus ADM solution of example 1 with step size of 0.2

X	Exact Solution	Solution with ADM	Absolut Error E_A
0.2	2.556141148067667	2.556141148067667	0.000000000000000
0.4	3.010377014931722	3.010377014931722	0.000000000000000
0.6	3.344598650004463	3.344598650004463	0.000000000000000
0.8	3.545481691392899	3.545481691392899	0.000000000000000
1.0	3.605017566159969	3.605017566159969	0.000000000000000
1.2	3.520832766855026	3.520832766855026	0.000000000000000
1.4	3.296283475765863	3.296283475765862	0.000000000000001
1.6	2.940321764521938	2.940321764521938	0.000000000000000
1.8	2.467138703248411	2.467138703248412	0.000000000000001
2.0	1.895598607382760	1.895598607382760	0.000000000000000
2.2	1.248486976948079	1.248486976948079	0.000000000000000

Table IV: Exact versus ADM solution of example 1 with step size of 0.3

X	Exact Solution	Solution with ADM	Absolut Error E_A
0.3	2.797233598235231	2.797233598235231	0.000000000000000
0.6	3.344598650004463	3.344598650004463	0.000000000000000
0.9	3.593200665423779	3.593200665423779	0.000000000000000
1.2	3.593200665423779	3.520832766855026	0.000000000000003
1.5	3.133959363147569	3.133959363147569	0.000000000000000
1.8	2.467138703248411	2.467138703248412	0.000000000000001
2.1	1.579935890746906	1.579935890746906	0.000000000000000
2.4	0.551602110570962	0.551602110570962	0.000000000000000
2.7	-0.526004643332633	-0.526004643332633	0.000000000000000
3.0	-1.556624969021289	-1.556624969021289	0.000000000000000
3.3	-2.448196622247474	-2.448196622247475	0.000000000000001

Figure 1: Exact solution of problem 1

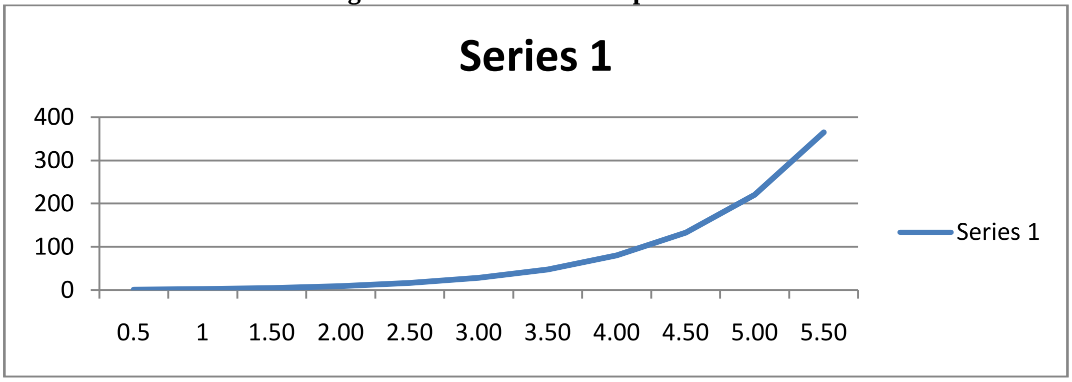


Figure 2: ADM solution of problem 1

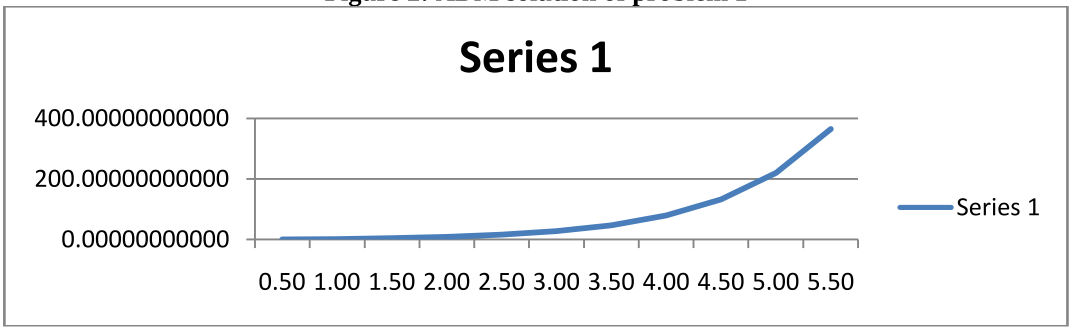


Figure 3: Exact solution of problem 2

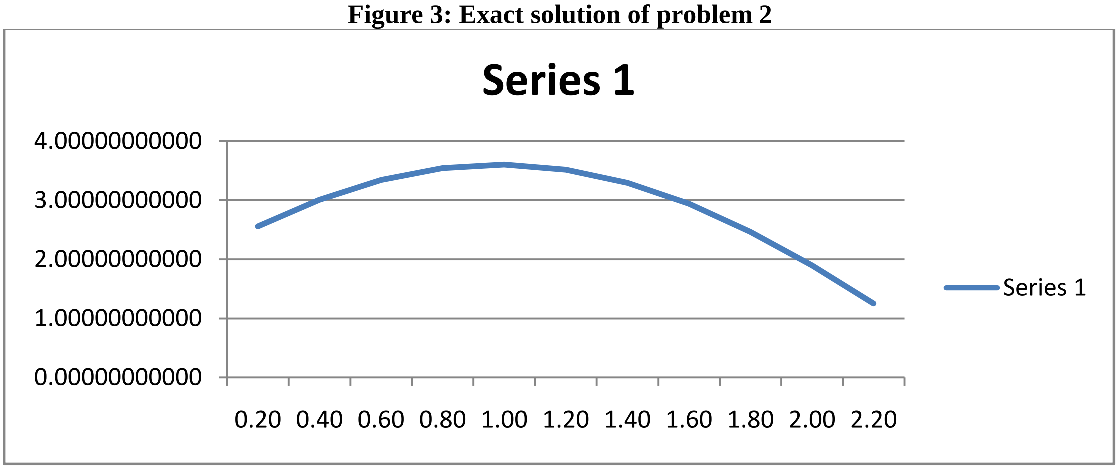
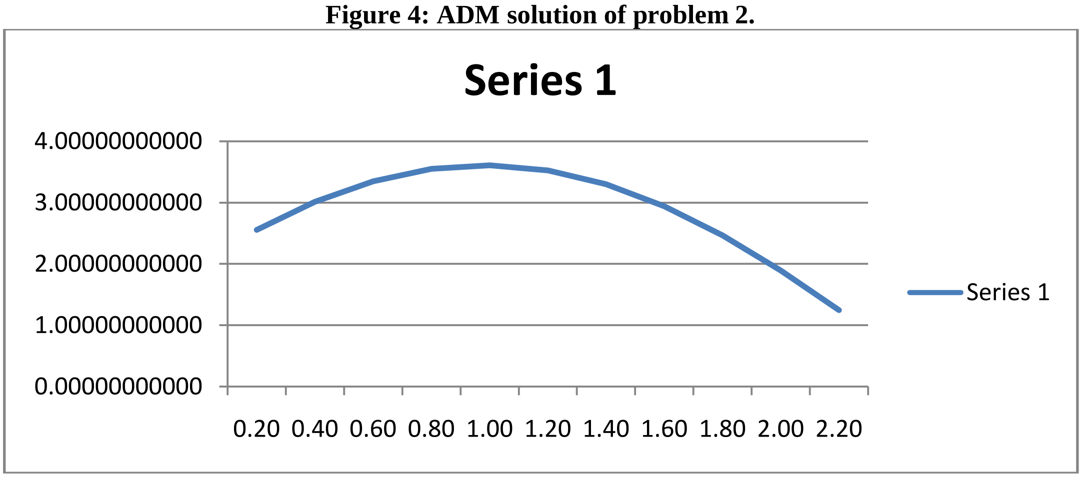


Figure 4: ADM solution of problem 2.



IV. DISCUSSION OF RESULTS

From the tables above, we can clearly see that the ADM is almost the same as the exact solution. Hence, the results presented here show that the method is reliable, accurate and converges very rapidly.

V. CONCLUSION

In this paper, we used the ADM to solve second orderautonomous and non-autonomous ordinary differential equations. Problem 1 is a non-autonomous ordinary differential equation while problem 2 is an autonomous ordinary differential equation. The ADM generates its solutions in the form of series and the round off errors inherited by taken a finite series from the infinite series. We observed that better accuracy can be obtained by accommodating more terms from the decomposition series and the solutions presented problems is stable and consistent in the interval $a \leq x \leq b$. We compared the numerical and the theoretical results and it shows that the ADM is almost the same as the exact solution. Hence, the ADM is very reliable, efficient, computational stable and promising. In our subsequent research, we shall examine the comparison with other existing methods.

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