On The Homotopy Theory of G - Spaces

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ABSTRACT.*The aim of this paper is to show that them ostelementary homo- topytheory of G-spaces is equivalent to ahomotopy theory of simplicialsets over **BG**,where Gisafixed group.Both homotopy theories are presented as Relative categories.We establish the equivalence by constructing a strict homotopy equivalence between the two relative categories.No Modelcategory structure is assumed on either Relative Category.*

I.   INTRODUCTION

A **G**-space is a simplicial set with the action of a group **G**. The category of **G**-spaces and **G**-equivariant simplicial maps is denoted by **G***Sp*. We establish an equivalence between the homotopy theory of **G**-spaces, T(**G***Sp*,Ω) and the homotopy theory T(**sSets**↓ **BG**,Σ). Both homotopy theories are presented as relative categories. The notation above is meant to distinguish the homotopy theory from the relative category presenting it. We denote the category of simplicial sets by **sSets**and we will assume the Kan model category structure on **sSets**, see [Qui67] and [Hov99]. The classifying space of the group **G** is denoted **BG**. The subcategory of weak equivalences, Ω, has the same objects as **G***Sp*and its morphisms are weak equivalences in **sSets**which preserve the group action. The subcategory of weak equivalences, Σ, has the same objects as **sSets**↓ **BG**. A morphism in Σ is an arrow of the comma category such that the arrow over the vertex **BG** is a weak equivalence in **sSets**i.e. its image under the geometric realization functor is a weak equivalence of topological spaces.

The goal of this paper is to provide a direct proof, in the context of relative categories, of a result on homotopy equivalence of simplicial categories obtained by *simplicial localizations*, see [DK80b, 3.3], [DK80a] and [DK80c], of the two relative categories (**G***Sp*,Ω) and (**sSets**↓ **BG**,Σ), which was announced in a note [DDK80, Theorem 2.1]. The proof indicated in [DDK80] is based on constructing suitable simplicial model category structures on the two relative categories (**G***Sp*,Ω) and (**sSets**↓ **BG**,Σ) and then establishing a homotopy equivalence, in the sense of [DK80b, 2.5], between the underlying *simplicial homotopy categories*, namely, the simplicial subcategories of cofibrant and fibrant objects of the two simplicial model categories in question.

   In this paper, we do not assume any model category structure. We have taken a very direct approach of writing a strict homotopy equivalence between the relative categories (**G***Sp*,Ω) and (**sSets**↓ **BG**,Σ) which establishes a homotopy equivalence of homotopy theories T(**G***Sp*,Ω) and T(**sSets**↓ **BG**,Σ). Finally, proposition 3.3 along with [BK11a, 2.3] and remark 3.1 show that our main result is an equivalent version of [DDK80, Theorem 2.1], thereby proving that theorem.[DDK80, Theorem 2.1] is a particular case of [Lur09, Theorem 2.2.1.2]. Jacob Lurie uses some sophisticated simplicial techniques to prove this generalized version of the theorem in [DDK80], as a homotopy equivalence of simplicial categories. In a future paper we plan on extending the arument of the proof of our main result to obtain an equivalence of homotopy theories of simplicial maps over an arbitrary simplicial set *B*, and a suitably defined homotopy theory of functors into the category of simplicial sets.

2. SETUP

        A homotopy theory is presented most naturally as a relative category. In this section we give a brief introduction to relative categories and explain the notion of weak equivalences in relative categories. A relative category is a pair (C,Γ), where C is an ordinary (small) category and Γ is a subcategory of C having the same set of objects as C. The morphisms of Γ will be called *weak equivalences* of C. A *functor of relative categories* is an ordinary functor which preserves weak equivalences. The category of all small relative categories and functors of relative categories,called **RelCat**, has been given a model category structure by Barwick and Kan see [BK11b]. Let **1** denote the category 0 → 1 having two objects and exactly one, nonidentity morphism. This category is treated as a relative category in which every morphism is a weak equivalences, namely (**1**, **1**).

Deftnition 2.1. A pair of functors of relative categories

$$F, G : (C, \Gamma) \to (C', \Gamma'$$

are *strictly homotopic* if there exists a functor of relative categories

$$H : C \times \mathbf{1} \to C,$$

such that $H(x,0) = F(x)$ and $H(x,1) = G(x)$, for all $x \in Ob(C)$. H is called a *strict homotopy* between F and G.

Notice that H is a natural weak eqivalence between F and G, H assigns to each $x \in Ob(C)$, a weak equivalence in C'. A morphism $f: x \to y$ in $Mor(C)$ is assigned a commutative pair of composite maps

$$H(f, 1) \circ H(x, 0 \to 1) = H(y, 0 \to 1) \circ H(f, 0)$$

Moreover, if f is a weak equivalence in C, then each arrow in the above equation is a weak equivaleces in C'.

Definition 2.2. A morphism of relative categories, $f : (C,\Gamma) \to (C',\Gamma')$, is a *strict homotopy equivalence* if there exists another morphism of relative categories, $\acute{f} : (C',\Gamma') \to (C,\Gamma)$ (called the inverse of f) such that the compositions $\acute{f}f$ and $f\acute{f}$ are strictly homotopic, see 2.1, to the identity maps of C and C' respectively.

2.1 The Homotopy category: The homotopy category of a relative category (C,Γ) is obtained by "formally inverting''' all morphisms in the subcategory Γ. Given two objects $X, Y \in Ob(C)$ and an integer $n \geq 0$, a *zigzag* in C from X to Y of length n is a sequence

$$X = C_0 \longrightarrow C_1 C_2 \cdots \qquad C_n = Y$$

of maps in C, each of which is either forward (i.e points to the right) or backward (i.e points to the left) and such a zigzag is called restricted if all the backward maps are weak equivalences or arrows in Γ. The homotopy category will then be the category HoC which has the same objects as C, in which, for every two objects $X, Y \in Ob(C)$, the hom-set $HoC(X,Y)$ is the set of equivalence classes of the restricted zigzags in C from X to Y, where two such zigzags are in the same class if one can be transformed into the other by a finite sequence of operations of the following three types and their inverses:
(i) omiting the identity map
(ii) replacing two adjacent maps which go in the same direction by their composition, and
(iii)   omitting two adjacent maps when they are the same, but go in the opposite direction
and in which the compositions are induced by the composition of the zigzags involved. If the category C is small, then the category HoC is also small.

2.2 A functorial construction of classifying spaces of groups: Let $\mathbf{G}$ be a discrete group. Let $\mathbf{G}[1]$ be the category with one object $*$ and $Hom\mathbf{G}(*, *)$ is isomorphic to $\mathbf{G}$. We claim that $N(\mathbf{G}[1])$ is the classifying space $\mathbf{BG}$. Let $\mathbf{G}[0,1]$ be the category with one object for each element of $\mathbf{G}$ and exactly one arrow between any two objects. Each object is both initial and final hence the nerve of $\mathbf{G}[0,1]$, denoted $\mathbf{EG} := N(\mathbf{G}[0,1])$ is a contractible simplicial set. Now we define the notion of an *action* of a group on a category An action of a group on a category is an assignment of an automorphism of the category to each element of the group. The action satifies the usual assiciativity and unit conditions. We define the (right) action of $\mathbf{G}$ on $\mathbf{G}[0,1]$ by assigning to each element $g \in \mathbf{G}$, a functor $\varphi_g : \mathbf{G}[0,1] \to \mathbf{G}[0,1]$ which is defined on objects by group multiplication as follows: $\varphi_g(x) = xg$ and since there is only one morphism between any two objects in $\mathbf{G}[0,1]$, there is only one way of defining the functor on arrows. Also, $\varphi_{g1g2} = \varphi_{g1} \circ \varphi_{g2}$ and if e is the unit element of the group then $\varphi_{ge} = \varphi_g$. This action of $\mathbf{G}$ on $\mathbf{G}[0,1]$ induces an action of $\mathbf{G}$ on $\mathbf{EG}$ This induced action is free and it is easy to check that $\mathbf{EG/G}$ is isomorphic to $\mathbf{BG}$.

A group homomorphism $\mathbf{G} \to \mathbf{H}$ induces a functor $\mathbf{G}[1] \to \mathbf{H}[1]$ and hence induces a map on their nerves. Let $Aut(\mathbf{G}[0,1])$ denote the monoid in *Cat* whose object is the category $\mathbf{G}[0,1]$ and morphisms are automorphisms of $\mathbf{G}[0,1]$, i.e. functors whose source and target is the category $\mathbf{G}[0,1]$ and which have an inverse. An action as defined above is uniquely determined by a functor $\mathbf{G}[1] \to Aut(\mathbf{G}[0,1]))$.

3. THE EQUIVALENCE OF HOMOTOPY THEORIES

The main result of this paper is presented in this section. We will prove the equivalence of the two homotopy theories by constructing a strict homotopy equivalence between the two relative categories presenting the homotopy theories. Given a simplicial set S, the product space $S \times \mathbf{EG}$ has a $\mathbf{G}$ - action and $S \times_{\mathbf{G}} \mathbf{EG}$ is the quotient space of this action which is also the total space of a fibration of simplicial sets

$$q_S : S \times_{\mathbf{G}} \mathbf{EG} \to \mathbf{BG},$$

which will be called the *quotient map*. Any *G* - space can be viewed as a functor from the category **G**[1] into the category **sSets**. The homotopy colimit of a functor$X \in Fun($**G**[1],**sSets**$)$ is the following quotient space:

$$hocolim X = X \times_\mathbf{G} \mathbf{EG},$$

where the simplicial set *X* on the right side of the above equation is the image of the functor*X*.

The following proposition is a key step in the proof of our main result 3.2:

Proposition 3.1. *The* **G***-space* $(X \times_\mathbf{G} \mathbf{EG}) \times_{qX} \mathbf{EG}$ *is isomorphic to the product* **G***-space* $X \times \mathbf{EG}$.

Proof. We prove this proposition by defining a morphism of **G**-spaces

$$K\colon X \times \mathbf{EG} \to (X \times_\mathbf{G} \mathbf{EG}) \times_{qX} \mathbf{EG}.$$

In degree *n*, this morphism is defined as follows:

$$(\mathrm{x}_n,\ e_n)\ \mapsto \big((x_n,\ e_n)\mathbf{G},\ e_n\big).$$

Using the freeness of the action of the group **G** on **EG**, it is easy to check that this map is an isomorphism of simplicial sets. Further it is not hard to see that this map preserves the **G**-action.

Theorem 3.2. *The relative categories* $(\mathbf{G}Sp,\Omega)$ *and* $(\mathbf{sSets}\downarrow \mathbf{BG},\Sigma)$ *are srtictlyhomotopy equivalent.*

Proof. We begin our proof by defining two functor of relative categories as follows:
The first

$$\mathrm{M}: (\mathbf{G}Sp,\Omega) \to (\mathbf{sSets}\downarrow \mathbf{BG},\Sigma)$$

is defined on objects as

$$\mathrm{M}(X) = q_X\colon X \times_\mathbf{G} \mathbf{EG} \to \mathbf{BG},$$

where $q_X\colon X \times_\mathbf{G}\mathbf{EG} \to \mathbf{BG}$ is the quotient map. A morphism $m\colon X \to Y \in \mathbf{G}Sp$induces a morphism on the quotient spaces

$$\underline{m}\colon X \times_\mathbf{G} \mathbf{EG} \to Y \times_\mathbf{G} \mathbf{EG}.$$

Moreover, $\underline{m}$is a morphism in **sSets**$\downarrow$ **BG**. The functor M is defined on morphisms as follows:
$\mathrm{M}(m\colon X \to Y) = \underline{m}\colon X \times_\mathbf{G} \mathbf{EG} \to Y \times_\mathbf{G} \mathbf{EG}.$

Clearly M is a functor of relative categories. A morphism$f\colon Y \to \mathbf{BG} \in Mor(\mathbf{sSets})$ uniquely determines a fibration$R(f)\colon R(Y) \to \mathbf{BG}$, where $R(Y)$ is simplicial set with an acyclic cofibration $Y \to R(Y)$ in the model category **sSets**. We define the second functor of relative categories N : $(\mathbf{sSets}\downarrow \mathbf{BG},\Sigma) \to (\mathbf{G}Sp,\Omega)$ as follows:
$$f\colon Y \to \mathbf{BG} \mapsto R(Y) \times_{R(f)} \mathbf{EG}$$

This is a functor of relative categories also. Now, the composite functors are defined on objects as follows:

$$\mathrm{N}(\mathrm{M}(X)) = (X \times_\mathbf{G} \mathbf{EG}) \times_{qX} \mathbf{EG}$$

and

$$\mathrm{M}(\mathrm{N}(f\colon X \to \mathbf{BG})) = q_{R(X)\times R(f)}\mathbf{EG}\colon (R(X) \times_{R(f)} \mathbf{EG}) \times_\mathbf{G} \mathbf{EG} \to \mathbf{BG}.$$

We first define the homotopy

$$\mathrm{N}\,\mathrm{M}\colon \mathbf{G}Sp \times \mathbf{1} \to \mathbf{G}Sp$$

On objects, it is defined as

$$h(X,0) = \mathrm{N}(\mathrm{M}(X)) = (X \times_\mathbf{G} \mathbf{EG}) \times_{qX}\mathbf{EG}, h(X,1) = X$$

and on morphisms it is defined as

$$h(m\colon X \to Y, id_0) = (X \times_\mathbf{G} \mathbf{EG}) \times_{qX} \mathbf{EG} \overset{\underline{m}\ \times_\mathbf{BG}\ \mathbf{EG}}{\Rightarrow} (X \times_\mathbf{G} \mathbf{EG}) \times_{qY} \mathbf{EG}$$

$$h(X \to Y, id_1) = X \to Y$$

$$h(X \to Y, \mathbf{1}) = (X \times_\mathbf{G} \mathbf{EG}) \times_{qX}\mathbf{EG} \to Y.$$

$(X \times_{\mathbf{G}} \mathbf{EG})$ is a principal **G**-space over **BG**. Let $q_X: (X \times_{\mathbf{G}} \mathbf{EG}) \to \mathbf{BG}$ be the quotient map. The **G** - map $h(X,1)$ is the following composition:

$$(X \times_{\mathbf{G}} \mathbf{EG}) \times_{q_X} \mathbf{EG} \cong X \times \mathbf{EG} \to X \to Y.$$

Clearly, if $X \to Y$ is a weak equivalence in $(\mathbf{G}Sp, \Omega)$, then the above map is also a weak equivalence in $(\mathbf{G}Sp, \Omega)$. Hence h is a weak equivalence preserving functor or $h \in Mor(\mathbf{RelCat})$.

Now we define the reverse homotopy $k: 1_{(\mathbf{sSets}\downarrow\mathbf{BG},\Sigma)} \Rightarrow MN$.

$$k: \mathbf{sSets}\downarrow \mathbf{BG} \times \hat{} 1 \to \mathbf{sSets}\downarrow \mathbf{BG}.$$

On objects, k is defined as

$$k(f: X \to \mathbf{BG}, 0) = f, k(f,1) = q_{R(X)}: R(X) \times_{R(f)} \mathbf{EG} \to \mathbf{BG}.$$

On morphisms, k is defined as follows:

$$k(f \to g, id_0) = f \to g,$$

$$k(f \to g, id_1) = R(X) \times_{R(f)} \mathbf{EG} \to R(Z) \times_{R(g)} \mathbf{EG},$$

where $f: X \to \mathbf{BG}$ and $g: Z \to \mathbf{BG}$ are objects of $\mathbf{sSets}\downarrow \mathbf{BG}$. For every $f \in Ob(\mathbf{sSets}\downarrow \mathbf{BG})$, there is a *zero morphism* obtained by the following composition: $0: R(X) \to * \to \mathbf{EG} \in Mor(\mathbf{sSets}\downarrow \mathbf{BG})$. Since the action of **G** on $R(X) \times_{R(f)} \mathbf{EG}$ is free, there is a weak equivalence of simplicial sets with contractible fibers

$$F: (R(X) \times_{R(f)} \mathbf{EG}) \times_{\mathbf{G}} \mathbf{EG} \to R(X). \text{ The morphism}$$
$$(id,0)$$
$$R(Z) \times_{R(g)} \mathbf{EG} \to (R(Z) \times_{R(g)} \mathbf{EG}) \times \mathbf{EG}$$

induces a morphism on quotient spaces

$$s: R(Z) \to (R(Z) \times_{R(g)} \mathbf{EG}) \times_{\mathbf{G}} \mathbf{EG}$$

which is a homotopy inverse of F. The following morphism determined by s

$$s/\mathbf{BG} : g \to qR(Z) \times_{R(g)} \mathbf{EG},$$

is a weak equivalence in $(\mathbf{sSets}\downarrow \mathbf{BG}, \Sigma)$.

We define the morphism $k(f \to g, \mathbf{1})$ by the following composite in $\mathbf{sSets}\downarrow \mathbf{BG}$:

$$k(f \to g, \mathbf{1}): f \to g \to^{s/\mathbf{BG}} q_{R(Z) \times_{R(g)} \mathbf{EG}}$$

Clearly, if $f \to g$ is a weak equivalence in $\mathbf{sSets}\downarrow \mathbf{BG}$, then the above arrow is also a weak equivalence in $\mathbf{sSets}\downarrow \mathbf{BG}$. Thus, k is a morphism of relative categories.

As mentioned in the introduction, the main goal of this paper is to provide a direct proof of [DDK80, Theorem 2.1]. Now we achieve this goal by proving that our main result is an equivalent version of [DDK80, Theorem 2.1]. More precisely, the following proposition along with our main result 3.2, [BK11a] and remark 3.1 prove [DDK80, Theorem 2.1].

Proposition 3.3. *The simplicial localization functor takes (strict) homotopy equivalences of relative categories to homotopy equivalences of simplicial categories in the sense of* [DK80b, 2.5].

3.1. **Future direction of this research.** The homotopy theory $T(\mathbf{G}Sp, \Omega)$ can be identified with a functor homotopy theory $\mathbf{sSets}^{h\mathbf{G}}$. The construction of this functor homotopy theory is elaborate and we will not describe it in this paper. The functor M in the proof above is derived from the homotopy colimit functor $hocolim: \mathbf{sSets}^{h\mathbf{G}} \to \mathbf{sSets}$ and the functor N in the proof above is a version of the homotopy pullback functor taking values in **G**-spaces

$$- \times^h_{\mathbf{BG}} \mathbf{EG} : T(\mathbf{sSets}\downarrow \mathbf{BG}, \Sigma) \to \mathbf{sSets}^{h\mathbf{G}}.$$

These two functors induce an equivalence between homotopy theories $\mathbf{sSets}^{h\mathbf{G}}$ and $T(\mathbf{sSets}\downarrow \mathbf{BG}, \Sigma)$. Let B be an arbitrary simplicial set, replacing **BG** by B we get homotopy theory $T(\mathbf{sSets}\downarrow B, \Lambda)$ whose weak equivalences are weak equivalences of simplicial sets over B. We claim the existence of another (relative) category $C(B)$ such that the homotopy theory $T(\mathbf{sSets}\downarrow B, \Lambda)$ is homotopy equivalent to a functor homotopy theory $\mathbf{sSets}^{hC(B)}$.

Remark. The definition of a **G**-space considered in [DDK80] is more generic in the sense that the group acting on a simplicial set could be a simplicial group. The argument of the proof our main result remains valid for this case also if we replace **EG** and **BG** by their equivalent versions for a simplicial group *G*, namely, *WG*

and $\overline{WG}$ respectively, see [May67]. We want to achieve our future goal mentioned in 3.1 by considering the action of ordinary monoids on simplicial sets. Therefore, writing our main result for the action of an ordinary group was more pertinent.

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