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# Orbital Transfer of a Rocket from Circular Orbit to Elliptical Orbit by Manuevering Its Thrust

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***ABSTRACT***
*A spacecraft or powered satellite,initially in circular orbit in vacuum is subjected to a thrust being a function of mass ,ie, a time-varying thrust so as to restore it to an elliptic orbit. The differential equations of motion of the spacecraft in polar form with reference to the centre of the Earth as pole are set up and solved. Ultimately the radial distance of of the satellite/ spacecraft is obtained as a function of mass. Finally this paper also deals with motion of a rocket under central repulsive thrust greater than the gravitational force.*

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## I. 1.INTRODUCTION

In space rocketry, change of orbit of sphere- shape powered satellite or any space craft is carried out by various techniques delineated in papers[3,4].In this article is dealt with a rocket which, initially in a circular orbit, is subjected to a variable thrust program to restore it in an elliptic orbit. Differential equations of motion of a rocket travelling with central time- varying repulsive force comprising repulsive thrust and inverse- square –law gravitational force are formed and solved leading to finding trajectory of the rocket and time taken as a function of true anomaly.

## II. FORMATION OF DIFFERENTIAL EQUATIONS

With as usual notations used in Central orbits[1,2], differential equations of motion in polar form for a spacecraft/rocket with reference to the centre of the Earth as the pole are

$$\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2} = \frac{\mu u^2 - T/m}{h^2u^2} \qquad (1)$$

where the thrust T and mass m are related at any time t , $V_E$ being the exhaust velocity of the rocket:

$$T = -V_E\,\frac{dm}{dt} \qquad (2)$$

$$r^2\frac{d\theta}{dt} = h \text{ (constant)} \qquad (3)$$

where r is the radial distance of the rocket from the centre of the Earth and $u=\frac{1}{r}$ , θ the angular displacement of the spacecraft from the initial line

$\theta = 0$ and μ the constant of proportionality for the universal law of gravity. For ellipticorbit the resulting force per unit mass ,ie, P needs to be inversely proportional to the square of the radial distance in (1) so that

$$\mu u^2 - \frac{T}{m} = ku^2$$

$$T = m(\mu - k)u^2;\ k = \text{constant} \qquad (4)$$

Equating (2) to (4) and using (3) one gets

$$V_E\,\frac{d(\log m)}{dt} = -(\mu - k)u^2 \qquad (5)$$

$$V_E\,\frac{d(\log m)}{d\theta} = \frac{k-\mu}{hV_E} = -\lambda(\text{constant}) \qquad (6)$$

Solving (6) by employing the initial conditions at t=0, $\theta = 0, m = m_0$ , $r=r_0=\frac{1}{u_0}$ and

$$\frac{du}{d\theta} = 0 \qquad (6.1)$$

the mass- variation law is obtained as

$$m = m_0 e^{-\lambda\theta} \qquad (7)$$

where $\frac{\mu-k}{hV_E} = \lambda. > 0$ (7.1)

## III. PROPELLANT CONSUMPTION

In order to determine the propellant consumption at time t, it is necessary to find mass m as a function of time t and in consequence of (7), θ as a function of time t. With this purpose is determined the elliptic path i.e. (r, θ), (θ, t) and finally (r, m) relations associated with the foregoing thrust programming. Using (7) the propellant mass consumption at any instant of time is given by

$$m_p = m_0 - m = m_0\left(1 - e^{-\lambda\theta}\right) \qquad (8)$$

4.VELOCITY AND ELLIPTIC PATH OF THE SPACECRAFT

In the light of (1), (3) and (4), one gets

$$\frac{d^2u}{d\theta^2} + u = \frac{k}{h^2} \qquad (9)$$

With this equation, initial conditions (6.1) and initial velocity $v_0$ perpendicular to the radius vector $r$, its velocity[1,2] v is given by

$$v^2 = h^2\left\{\left(\frac{du}{d\theta}\right)^2 + u^2\right\} = v_0^2 + 2\frac{k}{h^2}(u - u_0) \quad ,r_0 v_0 = h \qquad (9.1)$$

And the well-known solution[1,2] to equation(9 ) yields an elliptic path with the centre of the Earth as its one focus.

$$\frac{1}{r} = u = \frac{k}{h^2} + \left(\frac{1}{r_0} - \frac{k}{h^2}\right)cos\theta \qquad (9.2)$$

$$\frac{1}{r} = \frac{1}{l}(1 + e cos\theta) \qquad (10)$$

where its eccentricity e and semi-latus rectum l of the elliptic orbit exhibited in Figure 1 are given by

$$e = \frac{\frac{1}{r_0} - \frac{k}{h^2}}{\frac{k}{h^2}} = \frac{h^2}{r_0 k} - 1 < 1 \qquad (11)$$

Or, $\frac{h^2}{r_0 k} < 2$ and

$$l = \frac{h^2}{k} \qquad (12)$$

(θ, t) RELATION FOR ELLIPTIC ORBIT

This is done in case of an elliptic orbit in text book of Dynamics. But here it is done in a different way. Because of (3) and (9) the integral becomes

$$\frac{d\theta}{dt} = \frac{h}{l^2}(1 + cos\theta)^2 \qquad (13)$$

Observing (13), the following integral can be tackled in a straight forward method:

$$\int \frac{d\theta}{(1+ecos\theta)^2} = \int \frac{1}{e\sin\theta}\frac{d}{d}\left(\frac{1}{1+e\cos\theta}\right)d\theta \qquad \text{(integrating by parts)}$$

$$= \frac{1}{e}\left[\frac{1}{(1+e\cos\theta)sin\theta} + \int\frac{\cos\theta d\theta}{(1-\cos^2\theta)(1+e\cos\theta)}\right] + \text{constant} \qquad (14)$$

Now let us consider

$I = \int\frac{\cos\theta d\theta}{(1-\cos^2\theta)(1+e\cos\theta)}$, so that in the integrand can be split up into partial fractions as

$$\frac{\cos\theta}{(1-\cos^2\theta)(1+e\cos\theta)} = \frac{A}{1-\cos\theta} + \frac{B}{1+\cos\theta} + \frac{C}{1+e\cos\theta}$$

Or, $\cos\theta = A(1 + cos\theta)(1 + e\cos\theta) + B(1 - cos\theta)(1 + e\cos\theta)$

$+C(1 - \cos^2\theta)$ wherein substituting $\cos\theta = -1$, $\cos\theta = 1, \cos\theta = \frac{-1}{e}$, respectively, one gets

$$A = \frac{1}{2(1+e)},\ B = \frac{-1}{2(1-e)}, C = \frac{e}{1-e^2} \ ,$$

Substituting these in the above integral, it becomes

$$I = \int\left[\frac{e}{(1-e^2)(1+e\cos\theta)} + \frac{1}{2(1+e)(1-co\theta)} - \frac{1}{2(1-e)(1+\cos\theta)}\right]d\theta + \text{Const}$$

which can be easily evaluated with known formulae.

$$I = \frac{2e}{(1-e^2)^{\frac{3}{2}}}tan^{-1}{}^{\left(tan\frac{\theta}{2}\right)\sqrt{\frac{1-e}{1+e}}} - \frac{1}{2}\left\{\frac{1+\cos\theta}{(1+e)\sin\theta} + \frac{1-\cos\theta}{(1-e)\sin\theta}\right\} + \text{Const}$$

$$\text{Or, } I = \frac{2e}{(1-e^2)^{\frac{3}{2}}}tan^{-1}{}^{\left(tan\frac{\theta}{2}\right)\sqrt{\frac{1-e}{1+e}}} - \frac{1-e\cos\theta)}{(1-e^2)\sin\theta} + \text{Const} \qquad (15)$$

Employing (13),(14)and (15) together with the initial conditions (6.1),time to reach the true anomaly $\theta$ is given by

$$t = \frac{l^2}{h}\left[\frac{2}{(1-e^2)^{\frac{3}{2}}}tan^{-1\left(tan\frac{\theta}{2}\right)\sqrt{\frac{1-e}{1+e}}} + f(\theta)\right]$$

$f(\theta) = \frac{1}{e}\left[\frac{1}{(1+\cos\theta)sin\theta} - \frac{1-\mathrm{e}\cos\theta)}{(1-\mathrm{e}^2)\sin\theta}\}\right]$

$=\frac{1}{\mathrm{e}\sin\theta}\left(\frac{1}{1+\mathrm{e}\cos\theta} - \frac{1-ecos\theta}{1-\mathrm{e}^2}\right)$

$=\frac{-e^2(1-cos^2\theta)}{esin\theta\,(1-\mathrm{e}^2)(1+\mathrm{e}\cos\theta)} = \frac{-\mathrm{e}\sin\theta}{(1-\mathrm{e}^2)(1+\mathrm{e}\cos\theta)}$ so that

$$t=\frac{l^2}{h(1-e^2)^{\frac{3}{2}}}\left[2tan^{-1(tan\frac{\theta}{2})\sqrt{\frac{1-e}{1+e}}} - \frac{(1-\mathrm{e}^2)^{\frac{1}{2}}\mathrm{e}\sin\theta}{(1+\mathrm{e}\cos\theta)}\right] \quad (16)$$

when $\theta = \frac{\pi}{2}$ and $\theta = \pi$, the corresponding times are given by

$$(t)_{\frac{\pi}{2}} = \frac{l^2}{h(1-e^2)^{\frac{3}{2}}}\left[2tan^{-1\sqrt{\frac{1-e}{1+e}}} - \mathrm{e}(1-\mathrm{e}^2)^{\frac{1}{2}}\right] \quad (17)$$

$$(t)_{\pi} = \frac{\pi l^2}{h(1-e^2)^{\frac{3}{2}}} \quad (18)$$

Hence the time period of the elliptic orbit is

$$T^1=\frac{2\pi l^2}{h(1-e^2)^{\frac{3}{2}}} \quad (19)$$

which can also be verified by the fact that $b^2 = (1-e^2)a^2$

$$T^1 = \frac{\text{Area of thee ellipse}}{\text{The constant areal veloci ty}} = \frac{\pi ab}{\frac{1}{2}r^2\frac{d\theta}{dt}} = \frac{\pi a^3(\frac{b^2}{a})^2}{\frac{h}{2}b^3} = \frac{2\pi l^2}{h(1-e^2)^{\frac{3}{2}}}$$

where a and b are semi-major and semi-minor axes respectively such that the propellant mass consumption at any time, i.e., the mass-variation law with respect to time t can be obtained by combining (7) with(13). (t,r) relation can be found on lines[1]. (r,m) relation is obtained by eliminating θ between (7)and (9):

$$\frac{1}{r} = \frac{1}{l}\left[1 + \mathrm{e}\cos\left\{\frac{1}{\lambda}\log\frac{m_0}{m}\right\}\right] \quad (20)$$

Finally the value of constant k in relation (4) is acquired by use of the initial conditions (6.1) as

$$k=\mu - \frac{T_0 r_0^2}{m_0} \quad (21)$$

where $T_0$ is the initial thrust at time $t = 0$.

In this section the thrust per unit mass is considered inversely proportional to the radial distance and is repulsive from the centre of the Earth but is less than the universal gravitational acceleration so that overall acceleration of the spacecraft is directed towards the centre of Earth.

(r,t) RELATION FOR CENTRAL ATTRACTIVE THRUST

Recollecting equation (10) and employing half-angle formulae from Trigonometry,

$\mathrm{Cos}\theta = \frac{l-r}{re}$ , $\sin\theta=\frac{\sqrt{r^2e^2-(l-r)^2}}{re}$ , $\cos\theta = \frac{1-\tan^2\frac{\theta}{2}}{1+tan^2\frac{\theta}{2}}$

which give $\tan\frac{\theta}{2}=\sqrt{\frac{r(1+e)-l}{l-(1-e)r}}$ (22)

Substituting (22) in equation (16),one gets (r,t) relation

$$t=\frac{l^2}{h(1-e^2)^{\frac{3}{2}}}\left[2tan^{-1(\sqrt{\frac{r(1+e)-l}{l-(1-e)r}})\sqrt{\frac{1-e}{1+e}}} - \frac{(1-\mathrm{e}^2)^{\frac{1}{2}}\sqrt{r^2e^2-(l-r)^2}}{l}\right] \quad (23)$$

MOTION OF SPACECRAFT DUE TO REPULSIVE CENTRAL FORCE

However, if this 'inverse- square-law' acceleration, ie, if overall acceleration compatible with repulsive thrust is repulsive from the centre of the Earth then the path of rocket is neither a conic or nor a closed curve whose differential equation retaining all the parameters including the initial conditions as above become

$$\frac{d^2u}{d\theta^2} + u = -\frac{k}{h^2} \quad (24)$$

Hence the equation of the repulsive path is obtained as

$$\frac{1}{r} = u = -\frac{k}{h^2} + \left(\frac{1}{r_0} + \frac{k}{h^2}\right)cos\theta \quad (25)$$

and the velocity at time t is given by

$$v^2 = h^2\left\{\left(\frac{du}{d\theta}\right)^2 + u^2\right\} = v_0^2 - 2\frac{k}{h^2}(u - u_0) \quad (26)$$

Let us rewrite (23) as

$$u+\frac{k}{h^2}=\left(u_0+\frac{k}{h^2}\right)\cos\theta \le \left(u_0+\frac{k}{h^2}\right) \qquad (27)$$

which ratifies that $u \le u_0$ or $r \ge r_0$ at all time instants which suggests that radial distance r goes on increasing with the passage of time. As $u \to 0$, ie, $r \to$

$\infty$ in equation (27 ), $\cos\theta \to \frac{\frac{k}{h^2}}{u_0+\frac{k}{h^2}} < 1$ while $\theta < \frac{\pi}{2}$ . At $\theta = \frac{\pi}{3}$ , (27) gives $u = u_1 = \frac{1}{2}\left(u_0 - \frac{k}{h^2}\right)$; at $\theta = \frac{\pi}{6}$ , $u = u_2 = \frac{\sqrt{3}}{2}u_0 - \left(1-\frac{\sqrt{3}}{2}\right)\frac{k}{h^2}$ ; at $\theta = \frac{\pi}{4}$, $u =$
$u_2 = \frac{u_0}{\sqrt{2}} - (1-\frac{1}{\sqrt{2}})\frac{k}{h^2}$ which facilitate drawing of the path of the space vehicle as depicted in Figure 2.

$(\theta, t)$ RELATION FOR THE REPULSIVE PATH

Combining (3) with (27) is obtained

$$t=\int \frac{d\theta}{\{-\frac{k}{h^2}+(u_0+\frac{k}{h^2})\cos\theta\}^2} + \text{Const} = \frac{1}{(u_0+\frac{k}{h^2})^2}\int \frac{d\theta}{(\cos\theta - \frac{\frac{k}{h^2}}{u_0+\frac{k}{h^2}})^2} + \text{const} = \frac{1}{a^2}\int\frac{d\vartheta}{(\cos\theta-b)^2} + const = \frac{1}{a^2}\int\frac{1}{\sin\theta}\frac{d}{d}\left(\frac{1}{\cos\theta-b}\right)d\theta$$

$+\text{const} \qquad (28)$

where $u_0+\frac{k}{h^2} = a$ and $= \frac{\frac{k}{h^2}}{u_0+\frac{k}{h^2}} = b < 1 \qquad (29)$

Integrating (26) by parts is obtained

$$a^2 t = \frac{1}{\sin\theta}\left(\frac{1}{\cos\theta-b}\right) + \int\frac{\cos\theta d\theta}{(1-\cos^2\theta)(\cos\theta-b)} + \text{const} \qquad (30)$$

Let us separate the above integrand into partial fractions.

$$\frac{\cos\theta}{(1-\cos^2\theta)(\cos\theta-b)} = \frac{A}{1-\cos\theta}+\frac{B}{1+\cos\theta}+\frac{C}{\cos\theta-b}$$

where constants A,B,C can be evaluated :

$\cos\theta = A(1+\cos\theta)(\cos\theta-b) + B\big((1-\cos\theta)(\cos\theta-b)\big)$
$+C(1+\cos\theta)(1-\cos\theta)$

Putting -1,1,b for $\cos\theta$ in the above equation, we get

$$B=\frac{1}{2(1+b)},\quad A=\frac{1}{2(1-b)}\quad C=\frac{b}{(1-b)(1+b)}$$

$$I=\int\frac{\cos\theta}{(1-\cos^2\theta)(\cos\theta-b)} = \frac{1}{2(1-b)}\int\frac{d\theta}{1-\cos\theta}+\frac{1}{2(1+b)}\int\frac{d\theta}{1+\cos\theta}+\frac{b}{(1-b^2)}\int\frac{d\theta}{\cos\theta-b}+const$$

$$=\frac{1}{2(1-b)}\int\frac{(1+\cos\theta)d\theta}{(1+\cos\theta)(1-\cos\theta)}+\frac{1}{2(1+b)}\int\frac{(1-\cos\theta)d\theta}{(1+\cos\theta)(1-\cos\theta)}+\frac{b}{(1-b^2)}\int\frac{d\theta}{\cos\theta-b}+const$$

$$=\frac{-1}{2(1-b)}(\cot\theta+\operatorname{cosec}\theta)+\frac{1}{2(1+b)}(-\cot\theta+\operatorname{cosec}\theta)+I_1+\text{const} \qquad (31)$$

where $I_1 = \frac{b}{(1-b^2)}\int\frac{d\theta}{\cos\theta-b} = \frac{b}{1-b^2}\int\frac{d\theta}{\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}-b}$

$$=\frac{b}{1-b^2}\int\frac{\sec^2\frac{\theta}{2}}{\left(1-\tan^2\frac{\theta}{2}\right)-b\left(1+\tan^2\frac{\theta}{2}\right)}d\theta = \frac{2b}{1-b^2}\int\frac{1}{(1-b)-(1+b)\tan^2\frac{\theta}{2}}d\left(\tan\frac{\theta}{2}\right)$$

$$=\frac{2b}{(1-b^2)(1+b)}\int\frac{1}{\left(\frac{1-b}{1+b}\right)-\left(\tan^2\frac{\theta}{2}\right)}d\left(\tan\frac{\theta}{2}\right)$$ [ Using the formula $\int\frac{dx}{a^2-x^2}=\frac{1}{2a}\log\frac{a+x}{a-x}$]

$$=\frac{b}{(1-b^2)(1+b)}\sqrt{\frac{1+b}{1-b}}\log\frac{\sqrt{\frac{1-b}{1+b}}+\tan\frac{\theta}{2}}{\sqrt{\frac{1-b}{1+b}}-\tan\frac{\theta}{2}} = \frac{b}{(1-b^2)^{\frac{3}{2}}}\log\frac{\sqrt{\frac{1-b}{1+b}}+\tan\frac{\theta}{2}}{\sqrt{\frac{1-b}{1+b}}-\tan\frac{\theta}{2}}$$

Or, $$a^2 t = \frac{1}{\sin\theta}\left(\frac{1}{\cos\theta-b}\right)-\frac{1}{2(1-b)}(\cot\theta+\operatorname{cosec}\theta)+\frac{1}{2(1+b)}(\operatorname{cosec}\theta-\cot\theta)+\frac{b}{(1-b^2)^{\frac{3}{2}}}\log\frac{\sqrt{\frac{1-b}{1+b}}+\tan\frac{\theta}{2}}{\sqrt{\frac{1-b}{1+b}}-\tan\frac{\theta}{2}} \qquad (32)$$

Or, $$a^2 t = \frac{1}{\sin\theta}\left(\frac{1}{\cos\theta-b}\right)-\frac{\cos\theta+b}{(1-b^2)\sin\theta}+\frac{b}{(1-b^2)^{\frac{3}{2}}}\log\frac{\sqrt{\frac{1-b}{1+b}}+\tan\frac{\theta}{2}}{\sqrt{\frac{1-b}{1+b}}-\tan\frac{\theta}{2}}$$

$$=\frac{1-b^2-\cos^2\theta+b^2}{(1-b^2)(\cos\theta-b)\sin\theta}+\frac{b}{(1-b^2)^{\frac{3}{2}}}\log\frac{\sqrt{\frac{1-b}{1+b}}+\tan\frac{\theta}{2}}{\sqrt{\frac{1-b}{1+b}}-\tan\frac{\theta}{2}}$$

$$t=\frac{1}{a^2}\left[\frac{\sin\theta}{(1-b^2)(\cos\theta-b)}+\frac{b}{(1-b^2)^{\frac{3}{2}}}\log\frac{\sqrt{\frac{1-b}{1+b}}+\tan\frac{\theta}{2}}{\sqrt{\frac{1-b}{1+b}}-\tan\frac{\theta}{2}}\right] \qquad (33)$$

subject to the initial conitions(6.1)

(r,t ) RELATION

Finally we determine a relation between r and t for which we begin with equation (25), vis-a-vis applying circular functions of Trigonometry such as

$$\sin\theta=\sqrt{1-\cos^2\theta}=\frac{\sqrt{(u_0-u)(u_0+u+\frac{2k}{h^2})}}{u_0+\frac{k}{h^2}}$$

$$tan^2\frac{\theta}{2}=\frac{u_0-u}{u_0+u+\frac{2k}{h^2}} \qquad (\ u=\frac{1}{r}\ ) \qquad (35)$$

Putting (32) and (33) into (31) is obtained

$$t=\frac{1}{a^2(1-b^2)}\left[\frac{\sqrt{(u_0-u)\left(u_0+u+\frac{2k}{h^2}\right)}}{\left(u+\frac{k}{h^2}-b\left(u_0+\frac{k}{h^2}\right)\right)}+\frac{b}{(1-b^2)^{\frac{1}{2}}}\log\frac{\sqrt{\frac{1-b}{1+b}}+\sqrt{\frac{u_0-u}{u_0+u+\frac{2k}{h^2}}}}{\sqrt{\frac{1-b}{1+b}}-\sqrt{\frac{u_0-u}{u_0+u+\frac{2k}{h^2}}}}\right] \qquad (36)$$

MOTION OF THE SPACECRAFT WITH INVERSE SQUARE GRAVTATIONAL FORCE BALANCED BY THRUST AT ALL TIME INSTANTS

In this situation with no radial force and no transverse force the equations of motion can be rewritten as

$$\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2=0 \ \text{ and } r^2\frac{d\theta}{dt}=h \qquad (37)$$

which lead to

$$\frac{d^2r}{dt^2}=\frac{h^2}{r^3} \qquad (38)$$

Which can be solved subject to the initial conditions (6.1):

$$2\frac{dr}{dt}\left(\frac{d^2r}{dt^2}\right)=2\frac{h^2}{r^3}\frac{dr}{dt}$$

$$\left(\frac{dr}{dt}\right)^2=-\frac{h^2}{r^2}+\frac{h^2}{r_0^2} \qquad (39)$$

Hence the radial velocity is given by

$$\frac{dr}{dt}=h\sqrt{\frac{1}{r_0^2}-\frac{1}{r^2}} \qquad (40)$$

which is solved applying the initial conditions(6.1):

$$\frac{rdr}{\sqrt{r^2-r_0^2}}=\frac{h}{r_0}dt \qquad (41)$$

which leads to $\sqrt{r^2-r_0^2}=\frac{h}{r_0}t$ (42)

which gives the instantaneous radial distance r such that

$$r^2=r_0^2+\frac{h^2}{r_0^2}t^2 \qquad (43)$$

Now let us find the angular displacement at time t, for which putting (43) in (37),

$$\frac{d\theta}{dt}=\frac{h}{r_0^2+\frac{h^2}{r_0^2}t^2} \qquad (44)$$

which in consequence of (6.1) can be integrated as

$$\theta=\int\frac{hdt}{r_0^2+\frac{h^2}{r_0^2}t^2}=tan^{-1}\frac{ht^2}{r_0^2} \qquad \text{Or, } \tan\theta=\frac{ht}{r_0^2} \qquad (45)$$

Since there is no force acting on the rocket, it is in motion along a straight path because of the initial velocity $v_0$ of projection perpendicular to the initial radial distance $r_0$ such that $v_0=(r\frac{d\theta}{dt})_{t=0}$ and due to the second part of (37) $v_0r_0=h$. So relation (43) and (45) can be depicted as

$$r^2=r_0^2+v_0^2t^2 \qquad \tan\theta=\frac{v_0t}{r_0} \qquad (46)$$

Combining (45) with (43), the polar equation of the straight line path is obtained as

$$r=r_0\sec\theta \qquad (47)$$

However this aspect vis-à-vis relations (46) and (47) can be verified with the help of figure 3.(47) shows that at $\theta = 0$, r= $r_0$; at $\theta = \frac{\pi}{3}$, r=2$r_0$; at $\theta = \frac{\pi}{4}$, r=$\sqrt{2}\, r_0$ ; at $\theta = \frac{\pi}{6}$, r=2$(r_0)/\sqrt{3}$; as $\theta \to \frac{\pi}{2}$, r → infinity and t → infinity.The first part of equation can be rewritten as

$$\frac{r^2}{r_0^2} - \frac{t^2}{\frac{r_0^2}{v_0^2}} = 1 \qquad (48)$$

which represents (r,t) curve as a standard hyperbola with its centre at the centre of the Earth with conjugate axis =2 $r_0$ and transverse axis=2 $\frac{r_0}{v_0}$ where equation of its asymtode is r=$v_0$. Now let us find the mass- variation law in this case for which we in view of equation (2) write

$$T= -V_E \frac{dm}{dt} = \frac{\mu m}{r^2} \qquad (49)$$

Using first part of (46) in (49),

$$\frac{dm}{m} = -\frac{\mu dt}{V_E (r_0^2+v_0^2 t^2)} \qquad (50)$$

Integrating and applying the initial conditions (6.1) one gets

$$\log\frac{m}{m_0} = -\frac{\mu}{V_E}.\frac{1}{v_0 r_0} tan^{-1}\frac{v_0 t}{r_0}$$

$$m= m_0 e^{-\beta tan^{-1}\frac{v_0 t}{r_0}} \qquad (51)$$

where $\beta = \frac{\mu}{V_E}.\frac{1}{v_0 r_0}$ (51.1)

THRUST PROPORTIONAL TO THE CUBE OF THE RADIAL DISTANCE

Finally, in tandem with the above analysis, let us consider a situation wherein the thrust is is inversely proportional to the cube of the radial distance and is directed to the centre of the Earth ie

T =$\lambda u^3$

And as such the differential equation of motion of the rocket is

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2} + \frac{\lambda}{h^2}u \qquad (53)$$

Or, $\frac{d^2u}{d\theta^2} + u\left(1 - \frac{\lambda}{h^2}\right) = \frac{\mu}{h^2}$

In order to solve this equation in a simple method we are to substitute

$$u=U+\frac{\frac{\mu}{h^2}}{1-\frac{\lambda}{h^2}} \qquad (54)$$

$$\frac{d^2U}{d\theta^2} + U\left(1 - \frac{\lambda}{h^2}\right) = 0 \qquad (55)$$

which resembles equation of simple harmonic motion exhibiting a general solution

$$U=u-\frac{\frac{\mu}{h^2}}{1-\frac{\lambda}{h^2}} =A \cos(\sqrt{1 - \frac{\lambda}{h^2}}\ )\theta + Bsin(\sqrt{1 - \frac{\lambda}{h^2}}\ )\theta \qquad (56)$$

With the help of the initial conditions : $\theta = 0, u = u_0 = \frac{1}{r_0}, \frac{du}{d\theta} = 0$ (57)

and employing (54),we get

, A=constant = $\frac{1}{r_0} - \frac{\frac{\mu}{h^2}}{1-\frac{\lambda}{h^2}}$ and B=0 so that we acquire the equation of the path :

$$u=u_0 + \frac{\frac{\mu}{h^2}}{1-\frac{\lambda}{h^2}}\{1 - \cos\left(\sqrt{1 - \frac{\lambda}{h^2}}\ \right)\theta\} \qquad 1 - \frac{\lambda}{h^2} > 0 \qquad (58)$$

Multiplying (53) by $2\frac{du}{d\theta}.\frac{d^2u}{d\theta^2} + u\left(1 - \frac{\lambda}{h^2}\right).2\frac{du}{d\theta} = 2\frac{du}{d\theta}$

Integrating which one obtains the velocity v:

$$\frac{v^2}{h^2}=[(\frac{du}{d\theta})^2+u^2] = \frac{\lambda}{h^2}u^2 + \frac{2u}{h^2}\mu + c \qquad (59)$$

where constant c is to evaluated by use of the initial conditions (53):

$$c=\left(1 - \frac{\lambda}{h^2}\right)u_0^2 - \frac{2u_0}{h^2}\mu \qquad (60)$$

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