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Solution of a Portfolio Selection Problem Using Linear Model

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ABSTRACT*: A modified method of solution of portfolio selection problem in share market using multi-objective linear programming (MOLP) formulation is proposed in this paper. In the MOLP model construction, four important aspects of interest to an investor in share market viz. returns (short terms and long terms) received; risk of investment and liquidity of the invested shares are taken as the four objectives. The constructed problem is solved using the Min-max GP method and the Zimmermann Fuzzy method. The proposed methods of solution of the MOLP model are illustrated by a numerical example.*

KEYWORDS: Portfolio Optimization, Fuzzy Multi-objective Programming, return, risk, liquidity, Min-max GP.

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I. INTRODUCTION

While investing in the share market an investor primarily desires to maximize the expected return and at the same time minimize the concomitant risk and hence make a balance between the return and the risk. Apart from these two objectives the investor may want to optimize several other targets which he/she considers important. For example, liquidity of the shares is an important aspect, to be considered. Now the share market is unpredictable and fluctuation in the prices is a regular phenomenon. Thus, the cost prices of shares and its return is random in nature. So, the selection of portfolios, without proper planning and evaluation of the alternatives is a difficult task.

In 1952, Markowitz [1] first considered these aspects and combined probability theory and optimization theory to model the portfolio of investment in the share market. He proposed the mean-variance model for portfolio selection where the investment return was quantified as the expected value and risk as the variance and it is considered as one of the best methods for addressing such problems. Markowitz's model describes how an investor can select the optimum portfolio taking into consideration the trade-off between the expected return and the market risk. Markowitz mean-variance model may lead to erroneous conclusion, particularly when the security returns are asymmetric in nature. The existence of such asymmetric security return distribution was indicated in the works of Liu, et al. [2]; Yan and Li [3]; Guo,Q et al.[4]. To overcome the limitations of the mean-variance models, in 1959 Markowitz [5] proposed semi-variance in the place of variance as the measure of risk in portfolio selection. Several researchers worked on semi-variance and proposed their own models to minimize the semi-variance in a random environment. Few such works are Yan and Li [3], Guo,Q et al.[4], Mansini et al.[6], Ayub, et al. (2015)[7] etc,. Their works enriched the process of portfolio selection.

For application of mean-variance or mean semivariance method of optimal portfolio selection, the probability distribution of the returns is required. Now to apply the probability theory in the portfolio selection process, the decision-maker must be provided with a reasonably large size of statistical data pertaining to the performance of the securities. Many researchers proposed an alternative way of selecting portfolios based on expert's opinion regarding the subjective valuation of the security and their prospective returns. Their works can be broadly categorized into three ways: using fuzzy set theory Arenas et al. [8], Gupta et al. [9], Huang [10]; using possibility theory, Carlsson, et al. [11]; Zhang, et al. [12] and using credibility theory Huang [10]; Qin, et al. [13]. These methods are used in a situation where sufficient data regarding security returns is lacking. However fuzzy methods are also subjected to some drawbacks. When a fuzzy variable is used to represent the security returns, a paradox appears. Huang and Ying [14], Liu [15] proposed an alternative way to estimate a subjective expert's valuation of the security returns using uncertainty theory. Following this theory, many researchers subsequently worked on the problem of portfolio optimization. Some of such works are Yao [16], You [17].

The Multi-objective Linear Programming technique is also being used for solving portfolio selection problems. Arenas et al. [8] proposed a model that considered three criteria viz. return, risk and liquidity and applied fuzzy goal programming technique for the solution of the problem. Pankaj Gupta et. al. [9] considered asset portfolio optimization using fuzzy mathematical programming. They used semi absolute deviation for risk calculation. In the process, they considered not only yearly return for maximization but also maximized long term returns. Another method for optimum portfolio selection using linear programming under a crisp and fuzzy environment can be seen in the work of Hong-Wei Liu [18].

In the present paper a modified multi-objective linear programming (MOLP) model for portfolio selection has been considered. In the constructed model four objectives (three maximizing and one minimizing) have been set up. It is then solved by two methods viz. Zimmermann [19] technique under fuzzy environment and Min-max Goal programming, Flavell, et al. [20] which is akin to fuzzy method. To illustrate the proposed methods of solution of the MOLP model of portfolio selection, evaluation of the parameters involved in the model is necessary. This is done by collecting current data from Bombay Stock Exchange (BSE), India regarding monthly return, annual dividend, volume, market capitalization, current price, etc. offered to the shareholders by eighteen renowned companies. The parameters relevant to the problem are then calculated from the collected data pertaining to the selected eighteen companies and is placed in Table II. The portfolio selection problem is finally solved by the said methods using Lingo 18 software and the solutions are compared. With these objectives in mind the paper has been arranged as follows:

In section II, a modified linear model for the portfolio selection problem has been placed. In section III two methods of solution for the constructed MOLPP model have been proposed. The first one is a fuzzy method using Zimmermann technique and is discussed under section III.1. The second one is based on Min-max Goal Programming technique for solving MOLPP involving both maximizing and minimizing objectives and is placed in section III.2. Section IV deals with the solution by the said two methods of the constructed portfolio selection problem based on real data obtained from BSE over a period of ten years. The solutions obtained are also compared. Section V contains the conclusions about the findings of the present paper and in section VI some relevant references are placed.

II. FORMULATION OF PORTFOLIO SELECTION MODEL AS A MULTI-OBJECTIVE LINEAR PROGRAMMING PROBLEM

Let there are n possible ways of investments in the share market. Further let a potential buyer invests $x_i$proportion of his/ her wealth ini^{th} asset, i= 1 ,2, . . ., n. Then, x_1+x_2+...+x_n= 1. There are many aspects which concern an investor in the share market. For example, accrued returns on purchased shares, dividend earned, liquidity of the shares, concomitant risk, capital growth, security of principal amount invested, market availability and many others.

In this paper we considered four deciding aspects which mostly influence the financial status of gain or loss of an investor. The factors we considered are the return gained, dividend announced by the companies, liquidity and risk of investment. For constructing a linear portfolio selection model, the authors in [9], considered yearly average return per unit of each asset purchased for two different time periods viz. for one year and three years. Here we have taken an average of 3 years (respectively, 5 years) as short term (respectively. long term) return. Further average of 10 years return has been used to estimate the expected yearly return. The yearly dividend announced by the share issuing companies may be considered separately or it can be clubbed with the yearly returns, Markowitz [5]. From the historical records of a particular asset we can retrieve different parameters viz. volume, market capitalization, current price of each share leading to the calculation of liquidity.

Risk is measured either by calculating variance of the yearly returns or by calculating absolute semi variance below the expected return. In the present case as in [9] absolute semi variance below the expected return has been used as a measure of risk.

We use some notations as follows:

- $r_i = \frac{1}{10}\sum_{t=1}^{10} \quad r_{it}$= Expected yearly rate of return per unit of the i^{th} asset (estimated as the average yearly return over a period of 10 years).
- $r_i^1 = \frac{1}{3}\sum_{t=1}^{3} \quad r_{it}$ (Average yearly return per unit of the i^{th}asset over a period of 3 years).
- $r_i^2 = \frac{1}{5}\sum_{t=1}^{5} \quad r_{it}$ (Average yearly return per unit of the i^{th}asset (over a period of 5 years)

Where r_{it} denotes the return earned from the i^{th}asset for the t^{th}year of the period considered including the dividend announced by the company. It is calculated as follows:

r_{it}= return for i^{th}asset for t^{th}year

$$= \frac{closingpriceof \quad t^{th}year \; - \; closingpriceof \quad (t-1)^{th}year \; + dividendof \quad t^{th}year}{closingpriceof \quad (t-1)^{th}year} \quad \text{of the } i^{th} \text{ asset.}$$

- w_t= portfolio risk for the t^{th} year of all the assets.
- L_i= denotes the yearly average (calculated over a period of 5 years) liquidity per unit of i^{th}asset.

- **Objectives**

Now we set the following objectives to be fulfilled in respect of selecting the portfolio $x = (x_1, x_2, .., x_n)$.

- ***Short term return:***

The expected average return over a period of three years to be called short term return is expressed as
$f_1(x) = \sum_1^n r_i^1 x_i$ where $r_i^1 = \frac{1}{3}\sum_{t=1}^3 r_{it}$, $i = 1, 2, .., n$

- ***Long term return:***

The expected average return over a period of five years called long term return is expressed as
$f_2(x) = \sum_1^n r_i^2 x_i$ where $r_i^2 = \frac{1}{5}\sum_{t=1}^5 r_{it}$, $i = 1, 2, .., n$

- ***Liquidity:***

Liquidity means how easily a buyer can buy or sell the share without suffering a substantial loss. In other words, liquidity describes the degree to which an asset can be quickly bought or sold in the market at a price reflecting its intrinsic value. For an asset it may be measured by the turnover rate which is calculated by dividing the total number of shares traded over a period (i.e. volume) by the number of outstanding shares (i.e. the shares issued by the company) for the period. The higher the share turnover, the more liquid the shares of the company are.

So, to find the liquidity we need volume and total number of shares issued by the company. From the historical data of each company the monthly/daily record in respect of volume, market capitalization, current price could be obtained directly. From these records dividing the market capitalization by current price of each share we get the number of shares issued by a particular company.

$$totalnumberofsharesissuedbythecompany = \frac{marketcapitalization}{currentprice}$$

Now, by the definition we get, $liquidity = \frac{volume}{totalnumberofs\ \ haresissuedbyt\ \ hecompany}$

Using this formula month wise liquidity could be calculated from historical data for each company. Then taking an average of five years one can calculate yearly average liquidity of that share for a long term (5 years) basis.

Therefore, $f_3(x) = \sum_{i=1}^n L_i x_i$ [the yearly average liquidity for all the assets purchased over a period of 5 years] Where $L_i = \frac{1}{5}\sum_{t=1}^5 L_{it}$, i= 1 ,2, . . ., n, and L_{it} denotes the liquidity of the i^{th} asset for the t^{th} year of the period considered.

Another method adopted by Gupta, et.al [9] using possibility distribution is also used in this paper for the purpose of comparison with the average method. Here it is assumed that the yearly average liquidity L of any asset follow the trapezoidal probability distribution over the intervals of the form [$a-\alpha$, a, b, $b+\beta$]. So, the possibility distribution of L regarding it as a fuzzy variable is given by,

$$\mu(L) = \{1 + \frac{L-a}{\alpha} if a - \alpha \le L \le a 1 \qquad if a \le L \le b 1 - \frac{L-b}{\beta} if b \le L \le b + \beta 0 \qquad otherwise \tag{1}$$

The parameters a , b, α and β are to be evaluated from the historical record of average liquidity per month for the asset considered. Using Fuzzy extension principle by Zadeh [21], the crisp possibilistic mean value of the liquidity for the portfolio $x = (x_1, x_2, .., x_n)$ will then be calculated by

$$f_3(x) = \sum_{i=1}^n \left(\frac{La_i + Lb_i}{2} + \frac{\beta_i - \alpha_i}{6}\right) x_i$$

where La_i(respectively Lb_i) = left (respectively right) end points of the tolerance interval for the liquidity of the i^{th} asset. 'α_i' , 'β_i', are the left and right spread respectively, considering the liquidity follow trapezoidal possibilistic distribution as in (1) over the interval $[La_i - \alpha_i,\ La_i, Lb_i,\ Lb_i + \beta_i]$. In this method to calculate 'La_i', 'Lb_i', 'α_i' and 'β_i' the following procedure is followed. The monthly average liquidities for each company are arranged in ascending order and grouped in intervals of equal width. Then we find the intervals which contain the most of the recorded data. Now we find the midpoints of the first and last of the intervals containing most of the data in the array mentioned. These two midpoints give the left and right endpoints of the tolerance interval and denoted by 'La_i' and 'Lb_i' respectively. Lastly to find the left spread we find the difference between 'La_i'and the smallest liquidity of the array and denote it by 'α_i'. Similarly, the right spread 'β_i' is found by subtracting 'Lb_i' from the highest liquidity of the array.

- ***Risk:***

$f_4(x) = \frac{1}{T}\sum_{t=1}^{T} \quad w_t(x)$ where $w_t(x) = \frac{\left|\sum_{i=1}^{n} \quad (r_{it}-r_i)x_i\right| + \sum_{i=1}^{n} \quad (r_i - r_{it})x_i}{2}$

measures the absolute semi deviation of the returns earned from all the $n$assets i= 1 ,2, . . ., n, for the t^{th}year below their corresponding expected return r_i. The function $f_4(x)$ stands for the average absolute semi deviation of the returns over the time period T. For a fixed year t, let us partition the set {1,2,...n} in two disjoint subsets A and B such that $i \in A$ implies $r_{it} \geq r_i$ i.e. the return for the t^{th}year of i^{th}asset is at least equal to the expected returns of the asset and $i \in B$ implies $r_{it} < r_i$i.e. the return for the t^{th}year of the i^{th}asset is less than its expected return.

Then, $w_t(x) = \frac{\left|\sum_{i\in A} \quad (r_{it}-r_i)x_i\right| + \sum_{i\in A} \quad (r_i - r_{it})x_i}{2} + \frac{\left|\sum_{i\in B} \quad (r_{it}-r_i)x_i\right| + \sum_{i\in B} \quad (r_i - r_{it})x_i}{2}$

Therefore $w_t(x) = \sum_{i\in B} \quad (r_i - r_{it})x_i$, since$\left|\sum_{i\in A} \quad (r_{it} - r_i)x_i\right| + \sum_{i\in A} \quad (r_i - r_{it})x_i = 0$

Thus $w_t(x)$ gives a measure of risk in selecting the portfolio x, for the t^{th}year, covering all the assets in the sense that it yields a return below the expected return by the amount $w_t(x)$. If we calculate the risk over a period of T years, the average risk per year in selecting the portfolio $x$is represented by

$f_4(x) = \frac{1}{T}\sum_{t=1}^{T} \quad w_t(x) = \frac{1}{T}\sum_{i=1}^{T} \quad \left(\sum_{i\in B} \quad (r_i - r_{it})x_i\right)$, where $B \subset \{1,2,..,n\}$, $i \in B$implies $r_{it} < r_i$

This expression can also be written as $f_4(x) = \frac{1}{T}\sum_{i\in B} \quad (\sum_{t=1}^{T} \quad (r_i - r_{it})x_i)$ and the same is used for numerical calculation purpose.

- ***Constraints***

Our aim is to optimize the above mentioned four objectives subject to the following constraints, involving the variables $x_i,\ i = 1, 2, .., n$.

- x_1+x_2+...+x_n= 1. [Budget constraint, $x_i \geq 0$]
- $x_i \leq u_i y_i$ [upper bound constraint for investment in i^{th}asset]
- $x_i \geq l_i y_i$ [lower bound constraint for investment in i^{th}asset]

where $u_i,\ l_i$respectively denote the highest and lowest proportion of investment in the i^{th}asset and $y_i$is a binary variable defined by

$$y_i = \{1 \qquad if the investor invests in the i^{th} asset i.e. x_i \neq 0 0 \qquad otherwise$$

$\sum_{i=1}^{n} \quad y_i = l$[constraint representing the highest number assets included in the list.]

Clearly l can take one of the integral values $1, 2, .., n.$; so we restrict $l \in [1, n]$. In the solution process 'l' will be automatically determined. It may be mentioned here that in [9] the authors left it for the decision makers to choose the value of 'l'. But in our modified model, it will be system generated. This small change has a significant effect on the solution obtained and the same is discussed in the conclusion part.

Considering the objectives and the constraints detailed in section II, the following multi-objective linear programming problem (MOLPP) is proposed for the solution of the portfolio selection problem.

$$\text{Max } f_1(x) = \sum_1^n \quad r_i^1 x_i$$

$$\text{Max } f_2(x) = \sum_1^n \quad r_i^2 x_i$$

$$\text{Max } f_3(x) = \sum_{i=1}^{n} \quad L_i x_i$$

$$\text{Max } f_4(x) = \frac{1}{T}\sum_{t=1}^{T} \quad w_t(x)$$

subject to , (2)

$$x_1 + x_2 + \ldots + x_n = 1$$
$$x_i \geq l_i y_i$$

$$x_i \leq u_i y_i$$
$$u_i \in [0,1]$$
$$l_i \in [0,1]$$
$$\sum_{i=1}^{n} y_i = l$$
$$l \in [1, n]$$
$$y_i \in \{0,1\}$$
$$x_i \geq 0$$
$$i = 1,2, \ldots n$$

where $w_t(x)$, r_i , r_{it}, r_i^1, r_i^2, $L_i$are defined as in section II. The constant $'l'$ represents the number of non-zero variables in the portfolio and this will be determined in the solution process.

III. TWO MODELS FOR THE SOLUTION OF THE PORTFOLIO SELECTION PROBLEM UNDER FUZZY ENVIRONMENT

We propose to solve the modified portfolio selection problem modelled in (2) using Zimmermann fuzzy method and Min-max Goal Programming [GP] method.

III.1*Solution Using Zimmermann Method*:

We use Zimmermann's [19] technique for the solution of the multi-objective portfolio selection problem (2) under fuzzy environment. For this we first calculate the max/min values of the objectives separately subject to the given constraints and also note the corresponding solution in each case. This is done by solving four single objective LPPs taking one objective at a time out of $f_k(x)$, $k = 1,2,3,4$and subject to the constraints of (2) using Lingo 18.

Let the optimal values and the optimal solutions of the single objectives LPPs are given by

$$z_1^* = f_1(x) = f_1(x_1^*) z_2^* = f_2(x) = f_2(x_2^*)$$
$$z_3^* = f_3(x) = f_3(x_3^*) w^* = f_4(x) = f_4(x_4^*) \qquad (3)$$

Where X is the feasible space defined by the constraints of (2).

Here x_1^* , x_2^* , x_3^* and $x_4^*$are respectively the optimal solutions of the single objective LPPs. The first one is with $f_1(x)$ as objective subject to the constraints of (2) and so on. These max/min values of the maximizing /minimizing objectives are respectively used as their optimistic values.

Next, we fuzzyfy the objectives of the problem (2) as follows:

$f_1(x) \gtrsim z_1^*$, $f_2(x) \gtrsim z_2^*$, $f_3(x) \gtrsim z_3^*$ and $f_4(x) \lesssim w^*$

where the symbols '$\gtrsim$' and '$\lesssim$' respectively represents essentially greater than or equal to and essentially smaller than or equal to , which are respectively the fuzzified version of '$\geq$' and '$\leq$' respectively [22].

To construct membership functions of the fuzzy objectives defined above with the corresponding ideal values as their fuzzy goals, another set of objective values (pessimistic values) is required. This is obtained by using Luhandjula's [23] comparison technique. The technique is discussed as follows.

We prepare the Table I for calculating the values of the objectives at each of the points x_1^* , x_2^*, x_3^*andx_4^*which are respectively the optimal solutions of the individual objectives.

*Table I: **Calculation of Pessimistic values of the Objective***

Solution	Objectives			
	z_1	z_2	z_3	w_1
x_1^*	$z_1(x_1^*) = z_1^*$	$z_2(x_1^*)$	$z_3(x_1^*)$	$w(x_1^*)$
x_2^*	$z_1(x_2^*)$	$z_2(x_2^*) = z_2^*$	$z_3(x_2^*)$	$w(x_2^*)$
x_3^*	$z_1(x_3^*)$	$z_2(x_3^*)$	$z_3(x_3^*) = z_3^*$	$w(x_3^*)$
x_4^*	$z_1(x_4^*)$	$z_2(x_4^*)$	$z_3(x_4^*)$	$w(x_4^*) = w^*$

From the above table we calculate the pessimistic values $\widehat{z_1}$, $\widehat{z_2}$, $\widehat{z_3}$ and $\widehat{w}$ of the objectives as follows:

$$\widehat{z_1} = \text{Min}\ \{\ z_1^*,\ z_1(x_2^*)\ , z_1(x_3^*),\ z_1(x_4^*)\ \}$$
$$\widehat{z_2} = \text{Min}\ \{\ z_2(x_1^*),\ z_2^*\ ,\ z_2(x_3^*),\ z_2(x_4^*)\ \}$$
$$\widehat{z_3} = \text{Min}\ \{\ z_3(x_1^*),\ z_3(x_2^*)\ ,\ z_3^*,\ z_4(x_4^*)\ \}$$
$$\widehat{w} = \text{Min}\ \{\ w(x_1^*),\ w(x_2^*)\ ,\ w(x_4^*),\ w^*\ \}$$

The method explained above is capable of extension in more involved cases.

Now, returning to the solution of the problem (2), let $\widehat{z_1}$, $\widehat{z_2}$, $\widehat{z_3}$ and $\widehat{w}$ are respectively the pessimistic values of the objectives $z_1 \equiv f_1(x)$, $z_2 \equiv f_2(x), z_3 \equiv f_3(x)$ and $w \equiv f_4(x)$ obtained by using Luhandjula's technique explained above. Using these ideals(optimistic) and pessimistic values of the objectives, the linear membership functions of the fuzzyfied objectives with the ideal values as their respective fuzzy goals are constructed as follows.

$$\mu_{z_1}(x) = \begin{cases} 1 & if z_1 \geq z_1^* \\ \frac{z_1-\widehat{z_1}}{z_1^*-\widehat{z_1}} & if \widehat{z_1} \leq z_1 \leq z_1^* \\ 0 & if z_1 \leq \widehat{z_1} \end{cases} \qquad (4)$$

$$\mu_{z_2}(x) = \begin{cases} 1 & if z_2 \geq z_2^* \\ \frac{z_2-\widehat{z_2}}{z_2^*-\widehat{z_2}} & if \widehat{z_2} \leq z_2 \leq z_2^* \\ 0 & if z_2 \leq \widehat{z_2} \end{cases} \qquad (5)$$

$$\mu_{z_3}(x) = \begin{cases} 1 & if z_3 \geq z_3^* \\ \frac{z_3-\widehat{z_3}}{z_3^*-\widehat{z_3}} & if \widehat{z_3} \leq z_3 \leq z_3^* \\ 0 & if z_3 \leq \widehat{z_3} \end{cases} \qquad (6)$$

and

$$\mu_{w}(x) = \begin{cases} 1 & if w \leq w^* \\ \frac{\widehat{w}-w}{\widehat{w}-w^*} & if w^* \leq w \leq \widehat{w} \\ 0 & if w \geq \widehat{w} \end{cases} \qquad (7)$$

Now we use the optimality principle of Bellman and Zadeh [24]. It states that the fuzzy set 'decision' is a confluence of its fuzzy objectives and constraints. Thus, using the linear membership values of the fuzzy objectives given in (4) to (7), the Zimmerman fuzzy model, for solving the portfolio selection problem based on the multi-objective linear program (2) is given by,

$$\begin{aligned} &\max \lambda \\ &\text{subject to,} \\ &\lambda \leq \mu_{z_1}(x) = \frac{z_1 - \widehat{z_1}}{z_1^* - \widehat{z_1}} \\ &\lambda \leq \mu_{z_2}(x) = \frac{z_2 - \widehat{z_2}}{z_2^* - \widehat{z_2}} \\ &\lambda \leq \mu_{z_3}(x) = \frac{z_3 - \widehat{z_3}}{z_3^* - \widehat{z_3}} \\ &\lambda \leq \mu_{w}(x) = \frac{\widehat{w} - w}{\widehat{w} - w^*} \\ &\sum_{i=1}^{n} x_i = 1 \\ &x_i \geq l_i y_i \\ &x_i \leq u_i y_i \\ &u_i \in [0,1] \\ &l_i \in [0,1] \\ &\sum_{i=1}^{n} y_i = l \\ &l \in [1,n] \\ &y_i \in \{0,1\} \\ &x_i \geq 0 \\ &i = 1,2,\ldots n \\ &\lambda \in [0,1] \end{aligned} \qquad (8)$$

which is a confluence of the fuzzy goals and constraints [24].
Here, $\lambda = \{\mu_{z_1}(x),\ \mu_{z_1}(x), \mu_{z_1}(x),\ \mu_w(x)\ ,\ 1\}$, and **1** stands for the constant function having the value 1, for all $x \in X$, representing the membership function of each of the crisp constraints in (8). In (8), $l_i$and u_i are respectively the lower and upper bounds of $x_i \in [0,1]$ and $y_i$is a binary variable defined by

$$y_i = \{ 1 \quad if x_i \neq 0 \quad 0 \quad if x_i = 0$$

and the constant 'l'is determined in the solution process.

III.2 *Solution Using Min-max Goal Programming:*

We next state the solution procedure of the portfolio selection problem (2) using Min-max Goal Programming [GP] [20] technique. Min-max GP is an important method to solve MOLPP involving both maximizing and minimizing objectives. This method is akin to the fuzzy method of solution of an MOLPP. Now for solving MOLPP (2) using Min-max GP, we consider the following system,

$$\min d$$

subjectto,

$$f_k(x) + n_k - p_k = \omega_k\ ,\ k = 1, 2, 3, 4 \qquad (9)$$

$$\beta_k n_k + \gamma_k p_k \leq d$$

$$x \in X$$

Where X is the feasible space defined by the constraints of (2). Here d is the maximum weighted deviation between the achievement of the goals and their aspiration levels ; $\omega_k$is the specified aspiration level for the k^{th}objective function $f_k(x)$; $n_k(resp. p_k)$ is the negative ($resp.$ positive) deviation from the aspiration level of the objective $f_k(x)$; and β_k, $\gamma_k$are the non-negative weights attached to the deviation variables as per decision makers choice such that

$\sum_{k=1}^{4} \quad (\beta_k + \gamma_k) = 1$ and $n_k p_k = 0, \quad k = 1,2,3,4.$

For the maximizing objectives $f_k(x)$, $k = 1,2,3, \omega_k = \max_{x\in X} f_k(x) = z_k^*$ and $\omega_4 = w^* = \min_{x\in X} f_4(x)$. The values $z_k^*,\ k = 1,2,3$ and w^* are the ideal values of the objectives. Since the ideal values have been used as aspiration levels for the maximizing objectives, we must have $f_k(x) \leq z_k^*$ and hence $p_k = 0$ for $k = 1,2,3$. Similarly, for the minimizing objective $f_k(x) \geq w^*$, and so $n_k = 0$ for $k = 4$.

Next, we restrict the goal deviations to unit-less numbers, for this normalization of the deviation variables is necessary. This is done by dividing the deviation constraints $\beta_k n_k + \gamma_k p_k \leq d$in (9) respectively by $t_k = z_k^* - \widehat{z_k},\ k = 1,2,3$ and by $t_4 = \widehat{w} - w^*$. Here $z_k^*(k = 1,2,3)$ and $w^*$are the ideal values (optimistic values) of the objectives. Also $\widehat{z_k}, (k = 1,2,3)$ and $\widehat{w}$ are their pessimistic values.
Thus, we have the following modified system,

$$\min \underline{d}$$

subject to

$$f_k(x) + n_k = z_k^*\ ,\ k = 1,2,3.$$

$$\beta_k \frac{n_k}{t_k} \leq \underline{d}$$

$$f_4(x) - p_4 = w^*$$

$$\gamma_4 \frac{p_4}{t_4} \leq \underline{d} \qquad (10)$$

$$x \in X$$

Where X is the feasible space defined by the constraints of (2). The weights of $\beta_k,\ k = 1,2,3$ and γ_k, $k = 4$ are chosen by the decision maker such that $\sum_{k=1}^{4} \quad (\beta_k + \gamma_k) = 1$.
Here $\underline{d}$is the maximum normalized weighted deviation between the achievements of the goals and their aspiration levels. The linear program (10) can now be solved by using Lingo software 18.

IV. ILLUSTRATION OF THE PROPOSED METHODS OF SOLUTION FOR PORTFOLIO SELECTION PROBLEM

We illustrate here the proposed techniques detailed in (8) and (10) for the solution of the modified portfolio selection problem (2). For this, evaluation of the parameters appearing in (2) is necessary. Now to find the numerical values of these parameters, contemporary secondary data has been collected from BSE. Secondary data pertaining to 18 renowned companies over a period of last ten years (2009-2019) in respect of the yearly closing values, volume, annual dividend announced by the companies, market capitalization and corresponding current price of each asset has been collected from the BSE, India (cf. http://in.finance.yahoo.com; http://www.moneycontrol.com).

The data collected from all the selected 18 companies are then used to calculate the parameters viz. yearly return (r_{it}) ; average yearly return over a period of 3 years (r_i^1) ; and over a period of 5 years (r_i^2) ; expected return over a period of 10 years (r_i) , yearly average liquidity over a period of 5 years (L_i) , risk w_t for 10 years .

r_{it}= return from i^{th} asset for t^{th} year.

$$= \frac{closingpriceof\ \ t^{th}year\ -\ closingpriceof\ \ (t-1)^{th}year\ +dividendof\ \ t^{th}year}{closingpriceof\ \ (t-1)^{th}year}\ of the i^{th} year$$

$$r_i^1 = \frac{1}{3}\sum_{t=1}^{3} r_{it}$$

$$r_i^2 = \frac{1}{5}\sum_{t=1}^{5} r_{it}$$

$$r_i = \frac{1}{10}\sum_{t=1}^{10} r_{it}$$

$L_i = \frac{1}{5}\sum_{t=1}^{5} L_{it}$, $i = 1, 2, .., 18$.

Also for the calculation of semi-absolute deviation of r_{it}, below the expected return r_i , the expression $\sum_{t=1}^{10} (r_i - r_{it})$, $r_{it} \le r_i$, $i = 1, 2, .., 18$ is needed to be evaluated using the parameters defined above. All these parameters evaluated using the collected data are presented in Table II.

Table II : Calculation of Parameters Involved

Sl. No.	*Name of the Company*	r_i^1	r_i^2	r_i	L_i	$\sum_{t=1}^{10} (r_i - r_{it})$, $r_{it} \le r_i$
1	ABB	0.01343	0.10840	0.28454	0.00010261	1.6265
2	ALBK	0.03840	-0.04343	0.25431	0.000938784	3.04344
3	ASHOK LEY	0.06365	0.56078	0.50608	0.00644625	3.5907
4	BEL	-0.00317	0.42115	0.32762	0.004015911	2.9091
5	BHEL	0.05440	-0.02823	-0.01483	0.003626337	1.4639
6	BPCL	0.18490	0.33877	0.33928	0.003215464	1.0414
7	CIPLA	0.02074	0.13000	0.13762	0.002831758	1.2796
8	DR REDDY	0.00592	0.05240	0.27364	0.004880608	1.7261
9	INFOSYSTCH	0.13979	0.20197	0.31420	0.002423228	1.2224
10	ITC	0.14599	0.08687	0.23058	0.00131193	0.852
11	SIEMENS	0.02225	0.13989	0.26938	0.001436007	2.2113
12	TATA POWER	0.08568	0.00663	0.04803	0.00215592	1.125
13	TITAN	0.54460	0.40095	0.48541	0.00289167	2.0792
14	VOLTAS	0.34367	0.35746	0.51692	0.00431945	3.2290
15	WIPRO	0.07736	0.07427	0.28013	0.00094175	1.7043
16	RELIANCE	0.39284	0.27565	0.18114	0.00158601	1.0388
17	KOTAK	0.25506	0.29725	0.39623	0.00139047	1.5485
18	SBI	0.21735	0.16000	0.09756	0.00301628	1.1405

Therefore, using the values of the parameters displayed in Table II, the objectives of our stated model are respectively given by

Max$f_1(x) = \sum_{i=1}^{18} r_i^1 x_i$

$= 0.01343x_1 + 0.0384x_2 + 0.06365x_3 - 0.00317x_4 + 0.0544x_5 + 0.1849x_6 + 0.020745x_7 + 0.00592x_8 + 0.1398x_9 + 0.14599x_{10} + 0.02225x_{11} + 0.08568x_{12} + 0.5446x_{13} + 0.34367x_{14} + 0.07736x_{15} + 0.39284x_{16} + 0.25506x_{17} + 0.21735x_{18}$.

Max$f_2(x) = \sum_{i=1}^{18} r_i^2 x_i$

$= 0.1084\ x_1 - 0.04343x_2 + 0.56078x_3 + 0.42115x_4 - 0.02823x_5 + 0.33878x_6 + 0.13x_7 + 0.05240x_8 + 0.20197x_9 + 0.08687x_{10} + 0.13989x_{11} + 0.00663x_{12} + 0.40095x_{13} + 0.35746x_{14} + 0.07427x_{15} + 0.27565x_{16} + 0.29725x_{17} + 0.16x_{18}$.

Max$f_3(x) = \sum_{i=1}^{18} \quad L_i x_i$
$= 0.00010261x_1 + 0.000938784x_2 + 0.00644625x_3 + 0.004015911x_4 + 0.003626337x_5 + 0.003215464x_6 + 0.002831758x_7 + 0.004880608x_8 + 0.002423228x_9 + 0.00131193x_{10} + 0.001436007x_{11} + 0.00215592x_{12} + 0.00289167x_{13} + 0.00431945x_{14} + 0.00094175x_{15} + 0.00158601x_{16} + 0.00139047x_{17} + 0.00301628x_{18}$.

Max$f_4(x) = \frac{1}{10}\sum_{t=1}^{10} \quad w_t\,(x) = \frac{1}{10}\sum_{i=1}^{18} \quad (\sum_{T=1}^{10} \quad (r_i - r_{it})x_i)$

= (1/10) $(1.6265x_1 + 3.04344x_2 + 3.5907x_3 + 2.90906x_4 + 1.46392x_5 + 1.04139x_6 + 1.279593x_7 + 1.72606x_8 + 1.2224x_9 + 0.85201x_{10} + 2.2113x_{11} + 1.12499x_{12} + 2.079207x_{13} + 3.229027x_{14} + 1.70433x_{15} + 1.038776x_{16} + 1.548457x_{17} + 1.140486x_{18})$. (11)

IV.1 *Solution of the Portfolio Selection Problem*

Here we solve the portfolio selection problem as a MOLPP modelled in (2) and the objectives involved are explicitly represented in (11). We do it by two methods viz. Zimmermann fuzzy method and min-max GP method [20].

IV.1.1 *Solution of the Portfolio Selection Problem using Zimmerman fuzzy method*

For the solution of the portfolio selection problem modelled in (2) using Zimmerman fuzzy technique detailed in section III.1, we consider the system (8) and solve it by Lingo software 18.

The four objectives $z_1 \equiv f_1(x)$, $z_2 \equiv f_2(x), z_3 \equiv f_3(x)$ and $w \equiv f_4(x)$ in (8) are explicitly given in (11). Now, to affect the solution, we need the optimistic and pessimistic values of the objectives. Considering the objectives of model (2), given in (11), one by one together with the constraints of (2) four single objective LPPs are formed. We solve these four LPPs separately by Lingo software and obtain the following four solutions:

z_1^*= max z_1= 0.5444600, x_i= 0,i= 1,2,...,18; $i \neq 13$; $x_{13} = 1$
z_2^*= max z_2= 0.5607800, x_i= 0,i= 1,2,...,18; $i \neq 3$; $x_3 = 1$
z_3^*= max z_3= 0.00644625, x_i= 0, i= 1,2,...,18; $i \neq 3$; $x_3 = 1$
w^*= min w= 0.08520100, x_i= 0, i= 1,2,...,18;$i \neq 10$; $x_{10} = 1$

These optimal solution of the individual objective are respectively denoted by x_1^*, x_2^* , x_3^* , x_4^*. Next we calculate the pessimistic values $\widehat{z_1}$, $\widehat{z_2}$, $\widehat{z_3}$ and $\widehat{w}$ of the objectives using Luhandjula's comparison technique explained in section 3. For this we compute all the objective values at each of these four individual optimal solutions x_1^*, x_2^* , x_3^* , x_4^* and these are displayed in Table III. Thus, by Luhandjula's comparison technique the pessimistic values of the objectives are given by

$\widehat{z_1}$= min {0.5446, 0.06365, 0.06365, 0.14599} = 0.06365.

Similarly, $\widehat{z_2}$= 0.08687, $\widehat{z_3}$= 0.00131193, and$\widehat{w}$ = 0.35907.

The calculations are shown in Table III.

Table III: Optimistic and Pessimistic values of the Objectives

Solution	*Value of the Objectives*			
	z_1	z_2	z_3	w
x_1^*	0.544460 =z_1^*	0.40095	0.002891665	0.2079207
x_2^*	**0.06365** = $\widehat{z_1}$	0.560780 = z_2^*	0.00644625	**0.35907**
x_3^*	**0.06365**	0.5607800	0.00644625 = z_2^*	**0.35907** = $\widehat{w}$
x_4^*	0.14599	**0.08687** = $\widehat{z_2}$	**0.00131193** = $\widehat{z_3}$	0.08520100 = w^*

Now substituting the values of z_1^*, z_2^* , z_3^* , w^* and $\widehat{z_1}$, $\widehat{z_2}$, $\widehat{z_3}$ and $\widehat{w}$ in system (8) we get the following Zimmermann fuzzy model for the solution of portfolio selection problem.

$$\max\ \lambda$$

subject to,

$$\lambda \le \mu_{z_1}(x) = \frac{z_1 - 0.06365}{0.5444600 - 0.06365} = \frac{z_1 - 0.06365}{0.48081}$$
$$\lambda \le \mu_{z_2}(x) = \frac{z_2 - 0.08687}{0.5607800 - 0.08687} = \frac{z_2 - 0.08687}{0.47391}$$

$$\lambda \le \mu_{z_3}(x) = \frac{z_3 - 0.00131193}{0.00644625 - 0.00131193} = \frac{z_3 - 0.00131193}{0.00513432}$$
$$\lambda \le \mu_w(x) = \frac{0.35907 - w}{0.35907 - 0.0852010} = \frac{0.35907 - w}{0.273869}$$
$$\sum_{i=1}^{n} x_i = 1$$
$$x_i \ge l_i y_i$$
$$x_i \le u_i y_i$$
$$u_i \in [0,1]$$
$$l_i \in [0,1]$$
$$\sum_{i=1}^{n} y_i = l$$
$$l \in [1,18]$$
$$y_i \in \{0,1\}$$
$$x_i \ge 0$$
$$i = 1,2,\ldots 18$$
$$\lambda \in [0,1] \qquad (12)$$

$$\lambda = \{\frac{z_1 - 0.06365}{0.48081}, \quad \frac{z_2 - 0.08687}{0.47391}, \quad \frac{z_3 - 0.00131193}{0.00513432}, \quad \frac{0.35907 - w}{0.273869}, \quad 1\}$$

The explicit expressions for $z_1 = f_1(x)$, $z_2 = f_2(x)$, $z_3 = f_3(x)$ and $w = f_4(x)$ are given in (11).
Solving the model (12) by Lingo 18 software the solution obtained is ,

$$\lambda = 0.5183789$$
$$z_1 = 0.3128917$$
$$z_2 = 0.3325349$$
$$z_3 = 0.003973453$$
$$w = 0.2171021$$
$$x_3 = 0.1551328$$
$$x_6 = 0.04947424$$
$$x_8 = 0.2585974$$
$$x_{13} = 0.5367955$$
$$x_i = 0, \text{ for other values of } i = 1,2,\ldots 18$$
$$l = 4 \qquad (13)$$

The obtained solution shows that for an investor, it is beneficial to purchase the shares of the four companies corresponding to the non-zero values of the decision variables in proportion to them for overall satisfaction of his/her objectives. The solution actually trade-offs among the interests of the objectives. This can be seen by comparing the individual optimal values of the objectives and their values obtained by the proposed methods. The ideal values of the objectives are z_1^*= 0.5444600, z_2^*= 0.5607800, z_3^*= 0.00644625, and w^*= 0.08520100. Whereas the solution yields the objective values z_1= 0.3128917, z_2= 0.3325349, z_3= 0.003973453, w= 0.2171021. The overall satisfaction of the decision maker in respect of achieving the target is 52% which is reflected by the value of λ.

IV.1.2 *Solution of the Portfolio Selection Problem using Min-max GP method*

For the solution of the portfolio selection problem modelled in (2) using Min-max GP technique detailed in section III.2, we consider the system (10). Now substituting the values of z_1^*, z_2^*, z_3^*, w^* and $t_k = z_k^* - \widehat{z_k}$, $k = 1,2,3$ and by $t_4 = \widehat{w} - w^*$ in the system (10) we get the following Min-max GP model for the solution of portfolio selection problem. Here the values of z_1^*, z_2^*, z_3^*, w^* and $\widehat{z_1}$, $\widehat{z_2}$, $\widehat{z_3}$, $\widehat{w}$ are the same as their corresponding values obtained in section IV.1.1, viz. z_1^*= 0.5444600, z_2^*= 0.5607800, z_3^*= 0.00644625 and w^*= 0.08520100, $\widehat{z_1}$= 0.06365, $\widehat{z_2}$= 0.08687, $\widehat{z_3}$= 0.00131193, and $\widehat{w}$ = 0.35907.

$$\min \underline{d}$$

subject to,

$$f_1(x) + n1 = 0.5444600$$
$$f_2(x) + n2 = 0.5607800$$
$$f_3(x) + n3 = 0.00644625$$

$$\beta_1 \frac{n_1}{0.48081} \leq \underline{d}$$
$$\beta_2 \frac{n_2}{0.47391} \leq \underline{d}$$
$$\beta_3 \frac{n_3}{0.00513432} \leq \underline{d}$$
$$f_4(x) - p_4 = 0.08520100$$
$$\gamma_4 \frac{p_4}{0.273869} \leq \underline{d}$$
$$x_1 + x_2 + \ldots + x_n = 1$$
$$x_i \geq l_i y_i$$
$$x_i \leq u_i y_i$$
$$u_i \in [0,1]$$
$$l_i \in [0,1]$$
$$y_1 + y_2 + \cdots + y_n = l$$
$$l \in [1,18]$$
$$y_i \in \{0,1\}$$
$$x_i \geq 0$$
$$i = 1,2, \ldots 18 \qquad (14)$$

The explicit expressions for $f_k(\boldsymbol{x})$ are given in (11). The weights $\beta_k \geq 0$, $k = 1,2,3$ and $\gamma_4 \geq 0$ are chosen so that $\beta_1+\beta_2+\beta_3+\gamma_4 = 1$. These weights are chosen by the decision maker according to his/her priority for the objectives. We can also take the null hypothesis of equality of all the weights. The solution obtained by Lingo software for some typical choice of the weights are displayed in Table IV.

Table IV: Solution obtained by Min-max GP method varying weights

Weights Chosen				*Solution obtained*					
β_1	β_2	β_3	γ_4	x_i	$\underline{d}$	z_1	z_2	z_3	w
0.25	0.25	0.25	0.25	x_3 = 0.1551328 x_6 = 0.04947424 x_8 = 0.2585974 x_{13} = 0.5367955	0.1204053	0.312892	0.332535	0.0039734	0.2171621
0.6	0.1	0.1	0.2	x_6 = 0.1063062 x_{13} = 0.765492 x_{16} = 0.1282018	0.0718216	0.486906	0.378277	0.0027587	0.1835496
0.4	0.4	0.1	0.1	x_3 = 0.252360 x_{13} = 0.74764	0.1008585	0.423226	0.441285	0.0037887	0.246065
0.15	0.15	0.5	0.2	x_3 = 0.4236923 x_8 = 0.3758604 x_{13} = 0.2004473	0.1266935	0.138357	0.337663	0.0051453	0.2586881
0.15	0.15	0.2	0.5	x_6 = 0.5064537 x_8 = 0.3066292 x_{13} = 0.1869171	0.1083192	0.197254	0.262588	0.003666	0.144532

From the Table IV of solution it is seen that for equal weights we obtain the same solution as that obtained by Zimmermann technique. Further it is observed that the maximum deviation of the achievement level from the targets is minimum for the choice of weights for the first, second, third and fourth objectives respectively as 0.4, 0.4, 0.1, 0.1. In this case the first two objectives reach closer to their respective ideal values.

IV.1.3 *Solution of Portfolio Selection Problem Considering Possibility Distribution of Liquidity Parameter*

Assuming that the liquidity (i.e. turnover rate) of the shares follow trapezoidal possibility distribution, we calculated the parameters namely $La_i, Lb_i, \alpha_i, \beta_i$, for the shares of each company based on the collected data. The process of calculation of liquidity has been discussed in section II. Now, we illustrate the method used to calculate the desired parameters based on collected data using frequency statistic method as in [9]. For this illustration purpose we consider the case of one company, say ABB, i= 1. In this method the monthly average liquidity for five years i.e. for consecutive 60 months are arranged in ascending order and are grouped in intervals of equal width. Then we find the intervals which contain the most of the recorded data. For the

company ABB, the intervals [0.00000650756, 0.00010650756], [0.0001065076, 0.00020650756], [0.00020650756, 0.00030650756], [0.00030650756,0.00040650756], contain the most of the data.
Now we find the midpoints of the 1st and last of these intervals. Here 0.0000565075 and 0.0003565075 are the mid points of [0.00000650756, 0.00010650756] and [0.00030650756,0.00040650756] respectively. These two midpoints will be denoted by 'La_1' and 'Lb_1' respectively and give the left and right endpoints of the tolerance interval. Here, La_1= 0.0000565075 and Lb_2= 0.0003565075.
Lastly to find the left spread we find the difference between 'La_1' and the lowest entry of array and denote it by 'α_1'. Similarly the right spread 'β' is found by subtracting 'Lb_1' from the highest entry of the array. In the array the lowest and highest entries are 0.0000060756 and 0.001414046477 respectively. So, for the present case, α_1= 0.0000565075 − 0.00000650756 = 0.00004999994 and β_1= 0.001414046477 − 0.0003565075 = 0.00105753898.
Now, mean liquidity for the company i= 1, is given by

$$L_1 = \frac{La_1 + Lb_1}{2} + \frac{\beta_1 - \alpha_1}{6} = \frac{0.0000565075 + 0.0003565075}{2} + \frac{0.00105753898 - 0.00004999994}{6} = 0.00037443067$$

Similarly, L_i's for other companies for $i = 2, 3, .., 18$ are calculated. All these calculated values are placed in Table V.

Table V: Mean Liquidity of the assets using Trapezoidal Possibilistic Distribution

Sl. No.	*Name of the Company*	La_i	Lb_i	α_i	β_i	L_i
1	ABB	0.0000565	0.0003565	0.0000500	0.0010575	0.0003744
2	ALBK	0.0003484	0.0019484	0.0002000	0.0021781	0.0014781
3	ASHOK LEY	0.00437761	0.0133776	0.0022500	0.0320726	0.0138480
4	BEL	0.0021367	0.0111367	0.001500	0.0120854	0.0084009
5	BHEL	0.0016423	0.0056423	0.001000	0.0297476	0.0084336
6	BPCL	0.0013395	0.0039395	0.000600	0.0137715	0.0048348
7	CIPLA	0.0012858	0.0048858	0.0004500	0.0068151	0.0041466
8	DR REDDY	0.0013205	0.0063205	0.0005000	0.0073536	0.0049628
9	INFOSYSTCH	0.0010420	0.0040420	0.0002500	0.0023924	0.0028991
10	ITC	0.0002950	0.001795	0.0002494	0.0044223	0.0017405
11	SIEMENS	0.0009570	0.002557	0.0007996	0.0171058	0.0044747
12	TATA POWER	0.0009820	0.002782	0.00029998	0.0045841	0.0025960
13	TITAN	0.0016020	0.006602	0.00124997	0.0238831	0.0078742
14	VOLTAS	0.0025018	0.0075018	0.0024999	0.0047495	0.0053767
15	WIPRO	0.0004632	0.0018632	0.00029997	0.0005505	0.0012050
16	RELIANCE	0.0006926	0.0024926	0.00014996	0.0014413	0.0018078
17	KOTAK	0.0004940	0.002494	0.00019963	0.0023033	0.0018446
18	SBI	0.0014890	0.005089	0.0005992	0.0093037	0.0047398

Using these values, the liquidity for the portfolio $x = \{x_1, x_2 \ldots x_{18}\}$ is given by
$f_3(x) = \sum_{i=1}^{18} L_i x_i$, where $L_i = \frac{La_i + Lb_i}{2} + \frac{\beta_i - \alpha_i}{6}$.
Therefore, the third (liquidity) objective for the models (8) and (10) in the present case is given by

$$\text{Max } f_3(x) = \sum_{i=1}^{18} L_i x_i$$
$$= 0.000374\, x_1 + 0.0014781\, x_2 + 0.0138480\, x_3 + 0.0084009\, x_4 + 0.0084336 x_5 + 0.0048348 x_6 + 0.0041466 x_7 + 0.0049628 x_8 + 0.0028991 x_9 + 0.0017405 x_{10} + 0.0044747 x_{11} + 0.0025960 x_{12} + 0.0078742 x_{13} + 0.0053767 x_{14} + 0.0012050 x_{15} + 0.0018078 x_{16} + 0.0018446 x_{17} + 0.0047398 x_{18}. \quad (15)$$

For the liquidity objective given in (15), the optimistic and pessimistic values are given by, $z_3^* = 0.013848$ and $\widehat{z_3} = 0.0017405$. Hence the membership function of $z_3$considering it as fuzzy is given by,

$$\mu_{z_3}(x) = \frac{z_3 - \widehat{z_3}}{z_3^* - \widehat{z_3}} = \frac{z_3 - 0.0017405}{0.013848 - 0.0017405} = \frac{z_3 - 0.0017405}{0.0121075} \quad (16)$$

With this changed third objective we first solve the model (8). For this we replace z_3= $f_3(x)$ and $\mu_{z_3}(x)$ of (12) by their new expressions respectively given in (15) and (16) above. The other objectives and their membership functions remain unchanged. Then we solve the model (12) by Lingo 18 software and get the following result given in (17).

$$\lambda = 0.5557973$$
$$z_1 = 0.3308829$$
$$z_2 = 0.3502679$$
$$z_3 = 0.00846982$$
$$w = 0.2068544$$
$$x_3 = 0.1657869$$
$$x_5 = 0.1568352$$
$$x_6 = 0.1587471$$
$$x_{13} = 0.5186309$$
$$x_i = 0, \text{ for other values of } i = 1,2,..,18 \tag{17}$$

Similarly, in the model (14) we replace the third objective by its new form (15) and also use its new optimistic and pessimistic values $z_3^* = 0.013848$ and $\widehat{z_3} = 0.0017405$. Then the model (14) is solved using Lingo 18 software. The result obtained are placed in Table VI.

Table VI: Solution obtained by Min-max GP method varying weights considering liquidity as a trapezoidal fuzzy variable.

Weights Chosen				*Solution obtained*					
β_1	β_2	β_3	γ_4	x_i	$\underline{d}$	z_1	z_2	z_3	w
0.25	0.25	0.25	0.25	x_3= 0.1657869 x_5= 0.1568352 x_6= 0.1587471 x_{13}= 0.5186309	0.1110507	0.330883	0.350268	0.0084698	0.206854
0.6	0.1	0.1	0.2	x_6= 0.016075 x_{13}= 0.677323 x_{16}= 0.306601	0.0065105	0.4922879	0.3615335	0.0059654	0.1743526
0.4	0.4	0.1	0.1	x_3= 0.25236 x_{13}= 0.74764	0.100859	0.423226	0.441285	0.00938177	0.246065
0.15	0.15	0.5	0.2	x_3= 0.461944 x_5= 0.360558 x_{15}= .1774977	0.124408	0.1456824	0.3200382	0.001083546	0.2555586
0.15	0.15	0.2	0.5	x_3= 0.0330864 x_5= 0.4048128 x_6= 0.4006639 x_{13}= .1614369	0.1117898	0.186129	0.207591	0.00708053	0.1464325

From Table VI, of a solution treating liquidity as a trapezoidal fuzzy variable, it is seen that for equal weights we obtain the same solution as that obtained by Zimmermann technique in the corresponding case. Here the minimum deviation of the achievement levels from their respective targets occurs for the choice of the weights 0.6, 0.1, 0.1, 0.2 respectively for the first, second, third and fourth objectives. Also, if the decision maker emphasizes on the liquidity objective then the deviation is maximum. On the other hand, if he/she attaches most importance on the short term return objective then the deviation becomes minimum.

In the proposed modified model (2) for optimum portfolio selection, 'l' is to be determined in the solution process i.e. system generated. On the other hand, in [9] the authors kept it for the decision makers to fix the value. The advantage of the proposed modification may be noticed by analyzing Table VII. In it, both the solutions, (i) giving particular value to 'l' and (ii) keeping it open are placed.

Table VII: Comparison of the solutions, cases for given 'l' and system generated 'l'.

Choosing 'l'= 4	*system generated 'l'*
λ = 0.5141826	λ = 0.5557973
x_5= 0.1646425	x_3= 0.1657869
x_{13}= 0.8359575	x_5= 0.1568352
–	x_6= 0.1587471
–	x_{13}= 0.5186309
z_1= 0.3305463	z_1= 0.3308829
z_2= 0.1646425	z_2= 0.3502679

z_3= 0.00796597	z_3= 0.0084698
w= 0.1978274	w= 0.2068544

From Table VII, it is seen that setting 'l'= 4 in the problem we get only two non-zero variables and hence the constraint has not been satisfied in the solution process. On the other hand, in our proposed model four non-zero variables in the solution are obtained. Also, the overall satisfaction of the solution reflected by the value of λ is higher in the proposed model than fixing 'l'. Similar observations are observed by setting 'l'= 2, 3, 5, 10 etc.

V. CONCLUSION

The paper presents a modification of the MOLP formulation of the portfolio selection problem [9] and also proposes two fuzzy methods for the solution of the same. For evaluation of the parameters of the considered MOLP model, relevant data has been retrieved for eighteen eminent companies from the BSE, India. Four important objectives have been set for the problem, three of them viz. short term (3 years) return, long term (5 years) return and liquidity of the assets are maximizing while minimization of the risk in the form of absolute semi deviations of the returns below the expected return have been considered. In the formulation, dividends announced by the companies have been clubbed in the calculation of yearly returns of the shares. So, the dividend has not been taken as a separate objective. The solutions obtained by the two methods are compared. It is observed that, Min-max GP method (for equal weights) and Zimmermann's fuzzy method yield the same solution. Also, with the increase of weights attached to objectives in Min-max GP method, the corresponding values of the objectives improve. Further it is observed that considering liquidity of the shares as fuzzy, the overall satisfaction of the decision maker in respect of achieving the targets improves from 52% to 56% (in the first method of solution). Similar changes are also noticed in the second method of solution. As a future scope of study, cost of transactions may be considered as another objective and the parameters like return, dividend, liquidity etc. as fuzzy, type2 fuzzy.

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